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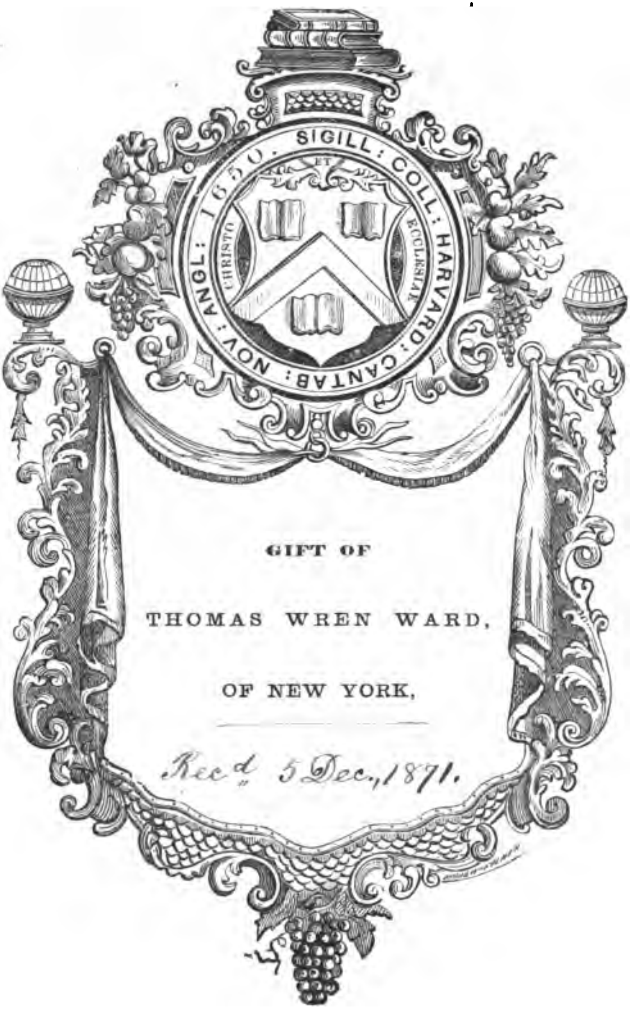
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OF FALSE DISCONTINUITY

WITH ILLUSTRATIONS FROM FOURIER'S THEOREM AND THE CALCULUS OF VARIATIONS.

BY

Michael Marlow Umfreville

M. M. U. WILKINSON, M.A.

RECTOR OF REEPHAM, NORFOLK,
AND FORMERLY FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

> "Intellectus humanus ex proprietate sua facile supponit majorem ordinem et æqualitatem in rebus quam invenit; et quum multa sint in natura monodica, et plena imparitatis, tamen affingit parallela et correspondentia et relativa quæ non sunt."

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PREFACE.

In this Essay I attempt to show that those problems, appertaining to the Calculus of Variations, to which the discussion of certain questions of maxima and minima naturally leads, admit, if of any, of an infinite number of solutions, and that of these solutions an infinite number are 'falsely discontinuous.' Mathematicians seem to me to use the word 'discontinuity' inappropriately, when they define it so that an unbroken curve coinciding throughout one portion of finite length with a circle, and throughout another portion of finite length with a straight line, is called 'discontinuous.' Such curves cannot with correctness be called discontinuous, because, however fortuitous and apparently irregular an unbroken line be, there corresponds to it a law such that a material point constrained by it shall construct the line. In this essay I define, and endeavour to employ as I define, the terms 'true discontinuity,' and 'false discontinuity.' I hope by the use of these words to adhere as closely as my views will permit to the definition of discontinuity as given by previous writers.

In the second chapter I endeavour to demonstrate Fourier's Theorem rigorously, and, in so doing, I think I have arrived at novel results by a method novel in many of its features. The subject I consider closely allied to the subject discussed in the rest of this essay. For an analysis of great power could replace all the methods of the Calculus of Variations by methods of Algebra and Differential Calculus operating upon series of sines and cosines, analogous to the method adopted in Art. 135.

A reader of this Essay will notice the analogy between the false discontinuity occurring in infinite trigonometrical series and that occurring in solutions of problems of Calculus of Variations.

I do not think that my method will fail to find the maxima and minima of a simple integral in any case where there is no failure of Taylor's Theorem for the expression under the integral sign.

In anticipation of criticism I would remark that, just as a function, $f(x, y)$, of two independant real variables should be said to have a minimum value when $x = x_1$, $y = y_1$, if $f(x_1, y_1)$ is never greater than $f(x_1 + h, y_1 + k)$, for all values of h^2, k^2, less respectively than some finite quantities, h_0^2, k_0^2, although $f(x_1, y_1)$ is greater than $f(x_1 + h', y_1 + k')$, for some values of h', k', otherwise chosen, in like manner $\int_{x_0}^{x_1} f(x, y, p)\, dx$ should be said to have a minimum value when $p = \phi(x)$, $y = \int_c^x \phi(x)\, dx$, if

$$\int_{x_0}^{x_1} \{f(x, y + \delta y, p + \delta p) - f(x, y, p)\}\, dx$$

is never negative whatever form δp, δy have, provided δp always less than Δp_0, and δy than $\Delta c_0 + \int_{x_0}^{x} \delta p dx$, where Δp_0, Δc_0 are assigned finite quantities.

Thus $x^2 y$ is said to have a minimum value when $x = 0$, $y = 1$, (for $h^2(1 + k)$ is always $+$ if k^2 is less than unity,) notwithstanding that $x^2 y = -1$, when $x = 1$, $y = -1$. In like manner

$$p = \Big|^{\alpha=0} \frac{\sin \alpha}{e^{x-x_1} - 2\cos\alpha + e^{-x+x_1}}, \quad y = \Big|^{\alpha=0} \int_{x_0}^{x} \frac{\sin \alpha\, dx}{e^{x-x_1} - 2\cos\alpha + e^{-x+x_1}},$$

should be said to render $\int_{x_0}^{x_1} \sqrt{1 + p^2}\, dx$ a minimum, as no finite values can be so assigned to δp, δy, as to make the integral less than $\left(x_1 - x_0 + \frac{\pi}{2}\right)$, notwithstanding that the integral equals $(x_1 - x_0)$ when $p = 0$.

The fact that such falsely discontinuous solutions, as the above, are solutions is a sufficient reason why they should be recognized. The fact that problems can be proposed which are solved, and can only be solved, by such falsely discontinuous solutions, is an additional reason why such falsely discontinuous solutions should be recognised in all cases.

The reader will observe that the solution of a problem in Calculus of Variations, obtained in accordance with the principles which I maintain, will comprise the solution of a problem much more general in its character, in which it is required that for particular values of the independant variable, the dependant variable and any of its successive differential coefficients should have assigned values. I have in the third chapter discussed some differential equations, partly to use the solutions in my examples, and partly to indicate that too much stress must not be laid upon current theories concerning how few constants the solution of a differential equation can contain.

If my views are, as I think they are, novel and peculiar, I hope mathematicians will be either convinced by my arguments, or think them sufficiently plausible to deserve refutation.

ERRATA.

PAGE	LINE	FOR	READ
9,	21,	$x=0, y$	$x_0=0, y_0$
13,	1,	51, 52	26, 27
14,	23,	when a is	when $\frac{ax}{\pi}$ is
25,	4,	$\cos\frac{2\theta}{s}$	$\frac{1}{2}\cos\frac{2\theta}{s}$
25,	4,	$\cos\frac{s\theta}{s}$	$\frac{1}{s}\cos\frac{s\theta}{s}$
25,	13,	$\cos s\delta$	$\frac{1}{2}\cos s\delta$
25,	13,	$\cos\frac{s\delta}{s}$	$\frac{1}{s}\cos s\delta$
28,	8,	x	n
41,	16,	$\int_0^a+\int_0^\pi$	$\int_0^a+\int_a^\pi$
81,	1,	r	r, finite or
81,	17,	r	r, finite or such
92,	2,	$OE=2a$,	$OE=4^{\frac{1}{2}}a$,
154,	5,	given	given, $(2c>x_1-x_0)$.
159,	2,	$y^2\sqrt{1+p^2}dx$	$\sqrt{y^2(1+p^2)\,dx}$

Omit line 9 on page 43.

Omit line 19 on page 80.

CHAPTER I.

OF CONTINUITY AND DISCONTINUITY.

1. In Time select in order n Epochs so that the intervals between each Epoch and the next Epoch in the selection are equal. An analogy or correspondence is manifest between Epochs so selected and n consecutive whole numbers. Suppose each interval of Time, selected as above, divided into m equal intervals, and compare the Epochs so obtained with the fractions whose numerators are whole numbers and denominators m. It will be seen that the same analogy is still observable. If further a particular Epoch be selected, and assigned to correspond to zero, and the convention be made that to positive whole numbers and fractions posterior Epochs shall correspond, and to negative prior, it will be obtained that one, and only one, Epoch corresponds to each particular vulgar fraction. It will also obtain that every particular Epoch can be distinguished from any different Epoch, however near it, in that a third Epoch, corresponding to a vulgar fraction, can be interpolated between them. Now every particular Ratio can be distinguished from any different Ratio, however near it, in that a third Ratio, commensurable with unity, can be interpolated between them. So the correspondence between Epochs and Ratios is so perfect that every Epoch may be represented by a Ratio.

2. The Human Mind is so constituted that it regards the flow of Time as continuous. That is to say Time is cognizable only under the condition that no two Epochs can be selected so near to each other that a third Epoch can not be interpolated between

them. This is not a definition but a test. The Succession of Ratios satisfies the same test. The Human Mind ought then to regard the succession of Ratios as continuous, although *a priori* it be disinclined so to do.

3. We shall call the particular Epoch selected to correspond to zero the Origin of Time, or the Original Epoch, and the Ratio which corresponds to any Epoch the Temporal ordinate, or, simply, the Time, of that Epoch.

4. In Space let any Point be selected and called the Origin of Space, or, simply, the Origin. Let three straight Lines be drawn through it, mutually at right angles to each other, and suppose three moving Points to move, one on each of these three straight Lines, and never to pass through the same Point twice. Then every Point on each of these three straight Lines will correspond to the Epoch at which the Point moving on the Line through it passed through it. For simplicity suppose that each moving Point was at the Origin of Space at the Origin of Time, and that each moving Point moves uniformly with the same velocity as the other two moving Points. Now every Point in Space corresponds to the regular parallelopiped of which it and the Origin are opposite corners, and the three straight Lines at right angles to each other through the Origin are edges. So every Point in Space corresponds to three Ratios which we shall call the three Spatial coordinates, or, simply, the Coordinates of that Point. It will be observed that the idea of continuity which attaches itself to the idea of Time, and which we are compelled to attach to the idea of Ratio, attaches itself also to the idea of Space.

5. Points in Space may be specified by other methods than that of the preceding article. Thus every Point in Space corresponds to the Sphere, right Cone, and Plane passing through it; the Centre of the Sphere being the Vertex of the Cone, and the Axis of the Cone constituting the edge of the Plane. It will be found, generally, that three Ratios are necessary to specify a Point in Space, but only one Ratio is necessary to specify an Epoch in Time.

6. Anything occupying at any Epoch a portion of Space and having other properties is called a Body. So that portion of a

Body which exists at a Point is itself a Body, and can be appropriately named a Material point. We recognize a Body as retaining its identity, notwithstanding that change of position corresponding to a change in Epoch, to which we refer when we speak of a Body's Motion; and the idea of continuity attaches itself to the Idea of Motion, in like manner as it attaches itself to the Ideas of Space and Time. When we say that the Motion of a Body is continuous we deny that the Body suffers perpetually annihilation and recreation, and we assert that between any two Epochs at which a constituent material Point of it has existed a third Epoch can be interpolated at which it did exist; and that, intermediate to any two positions which such Material point has occupied, a third position can be assigned which it did occupy.

7. The Conditions which obtain at any Point at any Epoch are, to a great extent, capable of being represented by Ratios just as Points and Epochs are. Now it is contrary to the nature of things that the same Conditions should both obtain, and not obtain, at the same Point at the same Epoch. So that, when the Time and the three coordinates are given, the Ratios which correspond to and represent the Conditions are determinate. This is true whether we consider events to be brought to pass in accordance with the regulations of any law or laws or otherwise. As, however, whatever the course of events has been or will be, it is always possible to recognize the existence of a law or laws such that the events may be supposed to have occurred in accordance with their regulations, it is convenient to select universal laws, and to suppose that events are carried on in accordance with their regulations. Such universal laws will make the Ratio which represents a Condition obtaining at a given Epoch at a given Point calculable from the Ratios which represent the Epoch and Point.

If a Ratio is determinate when certain other Ratios are specified, it is said to be a Function of the certain other Ratios, and the name Function is given to the law whereby the Ratio so determinate may be calculated from the certain other ratios.

8. Every actual Condition is a Function of the Temporal Ordinate and the three Spatial coordinates. As, however, we

know not all things, for us every actual Condition is a Function of the Time and the three Coordinates and such other Conditions as are requisite to supply our ignorance. A hypothetical Condition is a function of so many Ratios as, in accordance with the hypothesis, are essential to determine the Condition.

9. When Ratios are regarded as changing they are often called variables. When variables are Functions of other variables, the variables which are Functions of the others are said to be dependent, and the other variables, if not Functions one of another, are said to be independent. Thus the Condition obtaining at any Point at any Epoch is a dependent variable which is a Function of the Time and the three Coordinates as independent variables.

10. The Motion of a Material point being continuous, its distance from the Origin of Space at any Epoch will be a continuous Function of the Time. Conversely, when a Function of one independent variable is continuous, a Material point can be conceived as moving so that this Function, the Time being the independent variable, will express its distance from the Origin. It is impossible for such a Function to pass from one value to another without passing through intermediate values.

11. Suppose a Material point to move in one plane so that one Ordinate, x, is always equal to the Time, and another, y, to the distance of another Material point from the Origin. Then y will be a continuous Function of x. Now to every equation, $y=f(x)$, representing y to be a Function of x there corresponds a curve, the discontinuity or continuity of which will determine the discontinuity or continuity of the Function. If such curve cannot be conceived as traced by a moving Material point, it is discontinuous; if it can, it is continuous. In some respects this method of testing continuity is preferable to the method of Art. 10.

12. Accordingly a Function of one independent variable which, remaining finite, abruptly changes value, not passing through intermediate values, as the independent variable continuously passes through a particular value, is said to be discontinuous at that value. Also a Function is said to be discontinuous when it becomes

infinite; the reason being that we cannot conceive the possibility of a Material point moving to an infinite distance and reappearing. Also if a Function have real meaning for values of the independent variable on one side of a particular value, but not for values on the other, the Function is said to be discontinuous for that particular value of the independent variable.

13. For examples, in the equation,

$$y = +\frac{\sqrt{x^2}}{x},$$

y is always $= +1$ when x is $+$, and to -1 when x is $-$, and yet it never has any value different from one of these two; so it is discontinuous when $x = 0$.

$y = +\sqrt{x}$. In this equation y has real meaning when x is $+$, and not when x is $-$; so it is discontinuous when $x = 0$.

$y = \frac{1}{x}$, $y = \frac{1}{x^2}$. In both these equations y is infinite when $x = 0$, and is therefore discontinuous then.

14. As long as a Function of one independent variable is continuous it has a rate of increase or decrease, which rate corresponds to a Ratio, which Ratio is called the Differential coefficient of the Function. But when a Function is discontinuous we cannot then attach any meaning to its Differential coefficient; and all processes for obtaining it must be invalid from the nature of the case. For instance, when we say that if $y \equiv \sqrt{a^2 - x^2}$ (Art. 34), $\frac{dy}{dx} = \frac{-x}{\sqrt{a^2 - x^2}}$, we restrict x to values intermediate to $-a$ and $+a$; and when $x = a$, although we may take a $-$, we cannot take a $+$ increment for x. The Function, $(-a)^x$, where a is $+$, cannot be differentiated; for it is continuously discontinuous; no value however small can ever be given to δx so that $(-a)^{x'}$ is real for all values of x' between x and $(x + \delta x)$.

15. A Function of one independent variable may undergo a finite change while a change less than any change that can be assigned takes place in the independent variable and yet be not

discontinuous, if discontinuity be defined as in Article 12. It is inappropriate to say that a Function is discontinuous when such a change takes place, although the custom is sanctioned by eminent writers. We shall call discontinuity, as we have explained it, true discontinuity, and apply the term false discontinuity, as we shall now explain.

16. Suppose two Functions of x to differ from each other by less than any assignable Ratio for all values of x included between particular finite limits separated by a finite interval, and that this is not the case for all values of x. The Function which is not the simpler Function of the two is said to have false discontinuity at those particular values of x at which it ceases to have the value given by the simpler Function.

17. Thus, for instance, the Functions,

$$(a^2-x^2)^{\frac{1}{2}}, \qquad \Big|^{n=\infty} (a^2-x^2)^{\frac{2n+1}{4n+1}},$$

n being an integer, coincide in value for all values of x between the limits $-a$, $+a$; and the first, which is the simpler Function, has true discontinuity when $x=\pm a$, but the second has not. So the Function

$$\Big|^{n=\infty} (a^2-x^2)^{\frac{2n+1}{4n+1}}$$

has false discontinuity when $x=\pm a$.

18. Again, for instance, the Functions $\int_0^\infty \frac{\sin x\theta}{\theta}\,d\theta$, $\frac{\pi}{2}+x\delta$, where δ is less than any assignable quantity, coincide in value for all + finite values of x, and for no negative value of x, and the former Function is not the simpler Function of the two. The value of $\int_0^\infty \frac{\sin x\theta}{\theta}\,d\theta$, when x is indefinitely small, can be satisfactorily investigated as follows. It is axiomatic, or needs no demonstration, that, however great ∞ be, as x continuously passes through zero, from + to −, x must pass through the value $\frac{c}{\infty}$; now

$$\int_0^\infty \frac{\sin\frac{c\theta}{\infty}}{\theta}\,d\theta = \int_0^c \frac{\sin\phi}{\phi}\,d\phi, \text{ where } \phi = \frac{c\theta}{\infty},$$

so that the curve, $y = \int_0^\infty \frac{\sin x\theta}{\theta}\,d\theta$, coincides with the straight line $y = \frac{\pi}{2}$, when x is $+$, and with the straight line $y = -\frac{\pi}{2}$, when x is negative, the continuity of the two portions being maintained by the describing point making an infinite number of oscillations on the axis of y, the greatest maximum ordinate having the value $\int_0^\pi \frac{\sin\phi}{\phi}\,d\phi$. In accordance with our definition the Function $\int_0^\infty \frac{\sin x\theta}{\theta}\,d\theta$ has false discontinuity when $x = 0$.

19. Again, for instance, the Functions $+\sqrt{x^2}$, x, coincide in value for all $+$ values of x, and for no negative value of x, and the former Function is not the simpler Function of the two. So $+\sqrt{x^2}$ has false discontinuity when $x = 0$.

20. Again, let us examine the Functions

$$\frac{x}{x},\quad \frac{x}{+\sqrt{x^2}},\quad \Big|^{a=0} \frac{x}{+\sqrt{a^2+x^2}}.$$

The first is the simplest Function of the three, and is continuous, having always the value unity. The last is falsely discontinuous when $x = 0$, but not truly discontinuous, for when $x = ca$, $\frac{x}{\sqrt{a^2+x^2}} = \frac{c}{\sqrt{1+c^2}}$, which continuously passes through zero, as c continuously passes through zero. The middle Function has true discontinuity as x passes continuously through zero, passing from $+1$ to -1 without passing through any intermediate value.

21. The way in which our minds conceive the motion of a point to take place is that, in changing place from one position to another, it moves along a path continuously, and does not continually suffer annihilation at one point, and recreation at a point near to it. The notion which this word 'continuously' expresses is, that,

(calling the position of the point at any epoch 'the origin,') if the distance of the point from the origin at any other epoch is *d*, there is no distance intermediate in value to 0 and *d* such that the point has not been at that distance from the origin. Now, in like manner as we conceive a point to move, we can conceive a path to move. Such a conception involves the conception that every point in the path moves along a continuous path. The motion of the path consists in the motion of every point along the path, so moving that the path remains throughout unbroken. If the path break, we can no longer conceive it as a *single* path, but each fragment must be conceived separately as moving truly discontinuous from the rest. Although we must not suppose the path to break, yet we may suppose it to stretch or contract continuously to any extent. Truly discontinuous stretching or contraction involves, stretching the idea of creation, and contraction the idea of annihilation, and cannot therefore be conceived.

Calling the position of the path at starting 'the original path,' the path at any time has functions at its several points corresponding to the functions which were possessed by the original path. Now generally the change in any such function cannot be conceived as truly discontinuous: that is, it cannot pass from one value to another without passing through every intermediate value. When the time, and the position of the point on the path, are specified, all the functions of the path at that point have a fixed value. So that the change in the value of any function is itself a function of the time, such that it continuously changes in value from zero as the time changes in value continuously from zero.

Suppose that, at a time δt after the original time, the change that has taken place in a function, y, of the path at a point in it to be δy, then δy continuously changes from zero as δt continuously changes from zero, and $\frac{dy}{dt}$ is the initial value of $\frac{\delta y}{\delta t}$. Let δt be taken so small that $\frac{\delta y'}{\delta t}$ differ from $\frac{dy'}{dt}$ by less than any assignable quantity, where y' is any function, not then truly discontinuous, of the path at a point in it whatsoever, then δy is said to be the *variation* of the function y.

It must be observed that, although the initial value of δy must be zero, there is no occasion that the initial value of $\frac{\delta y}{\delta t}$ be finite, and that, in fact, we must expect that the case of $\frac{\delta y}{\delta t}$ having an initial infinite value will continually present itself. As there are passages in the works of writers on the Calculus of Variations which seem inconsistent with this, we will give a short illustration in the next Article.

22. Suppose a curve to move from the initial position, expressed by the equation,

$$x^2 + y^2 - 2ry = 0,$$

by rolling along the line, $y = 0$, uniformly, the centre moving with the velocity v.

At time t the point, whose initial position was x_0, y_0, finds itself in the position determined by the equations

$$y - r = (y_0 - r) \cos \frac{vt}{r} + x_0 \sin \frac{vt}{r},$$

$$x - vt = x_0 \cos \frac{vt}{r} - (y_0 - r) \sin \frac{vt}{r},$$

from which results

$$(x - vt)^2 + y^2 - 2ry = 0,$$

giving $$x - vt + (y - r) \frac{dy}{dx} = 0, \text{ so that}$$

$$\frac{dy}{dx} = - \frac{x_0 \cos \frac{vt}{r} - (y_0 - r) \sin \frac{vt}{r}}{(y_0 - r) \cos \frac{vt}{r} + x_0 \sin \frac{vt}{r}}.$$

Write $\frac{x_0}{y_0 - r} = \tan \theta$; then ($\theta = 0$ when $x = 0$, $y = 2r$, and)

$$\frac{dy}{dx} = - \frac{\tan \theta - \tan \frac{vt}{r}}{1 + \tan \theta \tan \frac{vt}{r}} = \tan \left(\frac{vt}{r} - \theta \right),$$

and $$\frac{d^2y}{dt\,dx} = \frac{v}{r}\sec^2\left(\frac{vt}{r} - \theta\right),$$

which is infinite when $t = 0$, and $\theta = \frac{\pi}{2}$, or $x_0 = y_0 = r$.

We observe that at that point $\frac{dy}{dx}$ is initially infinite, and that, as it changes in magnitude as the curve rolls on, there is no lesser infinite value so great but that $\frac{dy}{dx}$ has it at some time, the change at that point on the curve in $\frac{dy}{dx}$ being perfectly continuous. So that, although $\frac{d}{dt}\frac{dy}{dx}$ is initially infinite, $\frac{d}{dt}\frac{dy}{dx}\delta t$ can be made less than any assignable quantity by taking δt small enough, and $\delta\frac{dy}{dx}$ changes in value continuously from zero as δt changes in value continuously from zero, although $\frac{\delta\frac{dy}{dx}}{\delta t}$ is initially infinite.

23. As a point by its motion traces out a path or curve, so a path or curve by its motion will trace out a surface. And, in like manner as we conceive a point or a curve to move, so we can conceive a surface to move. The motion of a surface consists in the motion of every point upon the surface, so moving that the surface remains throughout unrent. The boundary of the surface will consist of initial and final positions of the generating path, together with the paths of the initial and final positions of the point which generated the path. Stretching and contraction may take place continuously to any extent. The moving surface has at any time functions at its several points corresponding to the functions which were possessed by the original surface at those points. When the time, and the position of the point on the surface, are specified, all the functions of the surface at that point have a fixed value. And (as in the case of a curve), if δy be the change that has taken place in a function, y, of the surface at a point in it, then δy continuously changes from zero as δt, the corresponding time since the original

time, changes; and if δt be taken so small that $\frac{\delta y'}{\delta t}$ differ from $\frac{dy'}{dt}$ by less than any assignable quantity for every function y', which is not then truly discontinuous, then δy is said to be the *variation* of the function y.

24. Every surface, capable of being conceived as a single surface, can be conceived as traced by motion as described in the last article. By its motion a *solid* can be conceived as traced out. In all these cases we have considered the moving concept is conceived as leaving something behind it, which something forms an element in the concept traced out. So, by the motion of a solid, we can conceive a solid traced out having conditions at each point of it continuously varying, as density, or any other condition of such a nature that, by reference to a unit of that condition, its magnitude can be measured. The definition we have given of the *variation* of a function applies in every conceivable case where the change can be figured to the mind as producible by motion merely, exclusive of creation or annihilation.

25. So we see that, whatever the function be, the variation of it is generally essentially a function not truly discontinuous, and one whose initial value is always zero, although its ratio to the initial value of the conceived change in the time corresponding to it, (the units of all magnitudes being finite,) may be either infinite, finite, or zero.

CHAPTER II.

OF THE REPRESENTATION OF TRUE AND FALSE DISCONTINUITY. FOURIER'S THEOREM.

26. We have seen that the function

$$\frac{+\sqrt{x^2}}{x}$$

is truly discontinuous when $x=0$. In like manner the function

$$\frac{a_1\sqrt{(x-b_1)^2}}{x-b_1}+\frac{a_2\sqrt{(x-b_2)^2}}{x-b_2}+\ldots+\frac{a_n\sqrt{(x-b_n)^2}}{x-b_n}$$

is truly discontinuous when $x=b_1, b_2, \ldots$, or b_n. At a point where a function becomes truly discontinuous its differential coefficient only applies in the continuous direction of the function (Art. 14). The differential coefficient of both the above functions is zero, $a_1, a_2, \ldots a_n$ being constant. We shall call such functions of x discontinuous constants.

27. As an illustration we will trace the curve (fig. 1),

$$y=\frac{x^2}{c}+\frac{a_1\sqrt{(x-b_1)^2}}{x-b_1}+\frac{a_1\sqrt{(x+b_1)^2}}{x+b_1}.$$

It coincides with the parabola,

$$y=\frac{x^2}{c}, \text{ as } x \text{ changes from } -b_1 \text{ to } +b_1.$$

If $x>+b_1$, it coincides with $y=\frac{x^2}{c}+2a_1$,

if $x<-b_1$, it coincides with $y=\frac{x^2}{c}-2a_1$,

as represented.

28. The method of Arts. 51, 52 enables us to break up any continuous function into a number of truly discontinuous bits, each bit being continuous. Other methods of doing the same thing may readily be invented, but none would be so simple. Thus, for instance, $\dfrac{\cos^{-1}\cos x}{x}$ has unity for its value as long as x is $+$ and $< \dfrac{\pi}{2}$, and -1 for its value when x is $-$ and $> -\dfrac{\pi}{2}$, the symbol $\cos^{-1}\phi$ meaning that root of the equation,

$$\phi = 1 - \frac{z^2}{2} + \frac{z^4}{\underline{|4}} - \ldots\ldots,$$

z being the unknown, which is intermediate to 0 and π, ϕ being < 1. As the method of Art. 26, 27 is very simple, and admits of universal application, we shall always suppose it the one employed whenever we have occasion to break up a continuous function into discontinuous bits.

29. There is, however, one case in which the formula of Art. 26 can advantageously be replaced. This case is when a discontinuous constant has to assume an infinite value for a particular value of x, and to continue to hold that constant value till x assumes another particular value.

The function $\dfrac{\pi}{4}\left\{\dfrac{\sqrt{x^2}}{x} - \dfrac{\sqrt{(x-b)^2}}{x-b}\right\}$ has zero for its value, except when x lies between 0 and b, when its value is $\dfrac{\pi}{2}$. So that the function

$$\tan\frac{\pi}{4}\left\{\frac{\sqrt{x^2}}{x} - \frac{\sqrt{(x-b)^2}}{x-b}\right\}$$

has zero for its value, except when x lies between 0 and b, when its value is infinite. And so discontinuous constants can be supposed to assume infinite values exactly as $\tan\dfrac{\pi}{2}$ has an infinite value.

30. It will be observed that the method of the preceding articles only affects the value of a function by altering it by a discontinuous constant. If then a falsely discontinuous function

can be found, which shall, between given limits for the independent variable, coincide with any assignable not truly discontinuous function, the perfect power of representation which analysis has will be made clear.

31. It is the object of a celebrated theorem of Fourier to shew how functions can be expressed by Trigonometrical series. A demonstration of this theorem will be given in order to shew that such series are falsely discontinuous. Some similar theorems concerning falsely discontinuous definite integrals will be added, which we shall have occasion to refer to in subsequent investigations.

32. Suppose that for all values of x between certain limits x_1, x_2, it is known that $f(x) = \phi(x)$. In Newton, Section I, Lemma 2, it is proved that $\int_{x_1}^{x_2} f(x)\,dx = \int_{x_1}^{x_2} \phi(x)\,dx$, supposing the functions do not become infinite for any value of x between the limits. It is erroneous to assume that $\frac{d}{dx}[f(x)] = \frac{d}{dx}[\phi(x)]$ between the same limits. It would follow from such an assumption that, in Newton, Section I, Lemma 2 (see fig. 2), the ultimate length of *aKbLcMdE* is equal to the length of the curve *abcdE*, which it is not. So also it would follow that, because

$$f(x) = f(x) + \Big|^{a=\infty} \int_{ax_0}^{ax} \frac{\sin x dx}{x},$$

provided x, x_0, be finite and $+$, $0 = \Big|^{a=\infty} \frac{\sin ax}{x}$, which is only true when a is an integer. So it is not true that the differential coefficients of vanishing functions are vanishing functions. All demonstrations are faulty in which it can be shewn that this assumption, or an assumption equivalent to it, occurs.

33. It follows from (32), that the power we possess of obtaining the differential coefficient of a function springs, not from our knowledge of the value which the function has for a given value of the independent variable, but from the mental concept of the law

expressed by the function. There must then be, in all laws which rest on experience, an assumption underlying them which does not rest on experience. Very generally a part of this assumption is that, out of a large number of laws all of which would equally give the result observed, that law must be selected which presents itself to the mind as the simplest concept.

34. When functions are identically equal, that is to say, are the same, not only in value but in conception, their differential coefficients will be identically equal also. We shall express the identity of two expressions by the symbol $\equiv$. So, $y \equiv f(x)$ will mean that y is identical with $f(x)$, that is, that y and $f(x)$ express the same function. It will follow from this that $\frac{dy}{dx} \equiv f'(x)$.

35. To take a case for illustration, if y be such a function of x that $\frac{d^2y}{dx^2} \equiv -y$, then $y \equiv A \cos(x+B)$, but we can only say

$$y = A \Big|^{s=\infty} \left\{1 - \frac{(x+B)^2}{2} + \frac{(x+B)^4}{\lfloor\underline{4}} - \dots + (-1)^s \frac{(x+B)^{2s}}{\lfloor\underline{2s}}\right\}.$$

The series gives

$$\frac{d^2y}{dx^2} + y = \Big|^{s=\infty} (-1)^s A \frac{(x+B)^{2s}}{\lfloor\underline{2s}};$$

and

$$\Big|^{s=\infty} \frac{d^{2s}}{dx^{2s}} (-1)^s A \frac{(x+B)^2}{\lfloor\underline{2s}} = (-1)^s A;$$

whereas

$$\Big|^{s=\infty} \frac{d^{2s}}{dx^{2s}} \left\{\frac{d^2}{dx^2}[A\cos(x+B)] + A\cos(x+B)\right\} = 0.$$

36. Suppose that, for all values of x between 0 and 2π,

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \dots + a_r \cos rx + \dots,$$

then

$$\int_0^{2\pi} \cos rx f(x)\, dx = a_r \int_0^{2\pi} \cos^2 rx dx,$$

for

$$\int_0^{2\pi} \cos rx \cos sx dx = \frac{1}{2}\int_0^{2\pi} [\cos(r-s)x + \cos(r+s)x]\, dx$$

$$= \frac{1}{2}\Big|_0^{2\pi} \left\{\frac{\sin(r-s)x}{r-s} + \frac{\sin(r+s)x}{r+s}\right\} dx,$$

which vanishes when r has a value different from s,

and
$$\int_0^{2\pi} \cos^2 rx dx = \frac{1}{2}\int_0^{2\pi} (1 + \cos 2rx)\, dx = \pi,$$

so that
$$\int_0^{2\pi} \cos rx f(x)\, dx = \pi a_r = \int_0^{2\pi} \cos r\theta f(\theta)\, d\theta.$$

From this it follows that, if $f_0(\theta)$ be some function such that, for all + integral values of r,

$$\int_0^{2\pi} \cos r\theta f_0(\theta)\, d\theta = 0.$$

(Now it should be assumed that there are such functions till the opposite has been proved,) then, $f(x)$ being perfectly general,

$$\pi f(x) + \pi f_0(x) = \sum_{r=0}^{r=\infty} \int_0^{2\pi} \cos r\theta f(\theta)\, d\theta \,.\, \cos rx.$$

We proceed to investigate the true expression for

$$\int_{-\pi}^{+\pi} \Phi(\theta) \left\{\frac{1}{2} + \Sigma \cos r\theta \cos r\beta\right\} d\theta,$$

and
$$\int_{-\pi}^{+\pi} \Phi(\theta) \,.\, \Sigma \sin r\theta \sin r\beta d\theta.$$

37. To prove that

$$\Bigg|^{a\gamma=\infty} \int_\gamma^\pi \frac{\sin a\theta}{\sin \frac{1}{2}\theta}\, d\theta = 0,$$

where a is +, and γ + and $< \pi$.

As $\sin a\theta$ changes sign only when $a\theta$ is a multiple of π,

let m be the integer =, or next >, $\dfrac{a\gamma}{\pi}$,

and ... n .... <, a.

Then $$\left[\int_\gamma^\pi = \int_\gamma^{\frac{m\pi}{a}} + \int_{\frac{m\pi}{a}}^{\frac{m+1}{a}\pi} + \dots\dots + \int_{\frac{n-1}{a}\pi}^{\frac{n\pi}{a}} + \int_{\frac{n\pi}{a}}^{\pi}\right] \frac{\sin a\theta}{\sin\frac{1}{2}\theta}\, d\theta.$$

These definite integrals alternate in sign, and, with the occasional exception of the second, each one is nearer zero than the preceding, for

$$\int_{\frac{r\pi}{a}}^{\frac{r+1}{a}\pi} \frac{\sin a\theta d\theta}{\sin\frac{1}{2}\theta} = \int_{\frac{r-1}{a}\pi}^{\frac{r\pi}{a}} \frac{\sin a\left(\theta+\frac{\pi}{a}\right)}{\sin\frac{1}{2}\left(\theta+\frac{\pi}{a}\right)}\, d\theta = -\int_{\frac{r-1}{a}\pi}^{\frac{r\pi}{a}} \frac{\sin a\theta d\theta}{\sin\frac{1}{2}\left(\theta+\frac{\pi}{a}\right)}.$$

Also $\displaystyle\int_{\frac{m-1}{a}\pi}^{\frac{m\pi}{a}} \frac{\sin a\theta}{\sin\frac{1}{2}\theta}\, d\theta$ is nearer zero than $\displaystyle\int_{\frac{m-1}{a}\pi}^{\frac{m\pi}{a}} \frac{\sin a\theta d\theta}{\sin\frac{m-1}{2a}\pi}$ or $\displaystyle\frac{(-1)^{m-1}2}{a\sin\frac{m-1}{2a}\pi}$, which ultimately vanishes, since $a\gamma$, and therefore m, is ultimately infinite.

So it follows that $$\Bigg|^{a\gamma=\infty} \int_\gamma^\pi \frac{\sin a\theta}{\sin\frac{1}{2}\theta}\, d\theta = 0.$$

38. To prove that, a being $+$, and γ being $+$ and $< \pi$,

$$\Bigg|^{a\gamma=\infty} \int_\gamma^\pi \frac{\cos a\theta}{\sin\frac{1}{2}\theta}\, d\theta = 0.$$

As $\cos a\theta$ changes sign only when $a\theta$ is an odd multiple of $\frac{\pi}{2}$,

let $(2m+1)$ be the odd integer $=$, or next $>$, $\frac{2a\gamma}{\pi}$,

and let $(2n+1)$ .... $<$, $2a$, then

$$\int_\gamma^\pi \frac{\cos a\theta d\theta}{\sin\frac{1}{2}\theta} = \left(\int_\gamma^{\frac{2m+1}{2a}\pi} + \int_{\frac{2m+1}{2a}\pi}^{\frac{2m+3}{2a}\pi} + \dots + \int_{\frac{2n-1}{2a}\pi}^{\frac{2n+1}{2a}\pi} + \int_{\frac{2n+1}{2a}\pi}^{\pi}\right) \frac{\cos a\theta d\theta}{\sin\frac{1}{2}\theta}.$$

These definite integrals alternate in sign, and (with the occasional exception of the second) each one is nearer zero than the preceding, for

$$\int_{\frac{2r+1}{2a}\pi}^{\frac{2r+3}{2a}\pi}\frac{\cos a\theta d\theta}{\sin\frac{1}{2}\theta}=\int_{\frac{2r-1}{2a}\pi}^{\frac{2r+1}{2a}\pi}\frac{\cos a\left(\theta+\frac{\pi}{a}\right)d\theta}{\sin\frac{1}{2}\left(\theta+\frac{\pi}{a}\right)}=-\int_{\frac{2r-1}{2a}\pi}^{\frac{2r+1}{2a}\pi}\frac{\cos a\theta d\theta}{\sin\frac{1}{2}\left(\theta+\frac{\pi}{a}\right)};$$

also $\displaystyle\int_{\frac{2m-1}{2a}\pi}^{\frac{2m+1}{2a}\pi}\frac{\cos a\theta d\theta}{\sin\frac{1}{2}\theta}$ is nearer zero than $\displaystyle\int_{\frac{2m-1}{2a}\pi}^{\frac{2m+1}{2a}\pi}\frac{\cos a\theta d\theta}{\sin\frac{2m-1}{4a}\pi}$

$$\text{or }\frac{(-1)^m 2}{a\sin\frac{2m-1}{4a}\pi},$$

which ultimately vanishes, since $a\gamma$, and therefore m, is ultimately infinite.

So it follows that $\displaystyle\Bigg|^{a\gamma=\infty}\int_{\gamma}^{\pi}\frac{\cos a\theta d\theta}{\sin\frac{1}{2}\theta}=0.$

39. To prove that $\displaystyle\Bigg|^{a\gamma=\infty}\int_{\pi}^{2\pi-\gamma}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}d\theta=0.$

$$\int_{\pi}^{2\pi-\gamma}\frac{\sin a\theta d\theta}{\sin\frac{1}{2}\theta}=-\int_{\pi}^{\gamma}\frac{\sin a(2\pi-\phi)}{\sin\frac{1}{2}(2\pi-\phi)}d\phi$$

$$=\int_{\gamma}^{\pi}\frac{\sin 2a\pi\cos a\phi-\cos 2a\pi\sin a\phi}{\sin\frac{\phi}{2}}d\phi=0,$$

by Arts. 37, 38.

40. To prove that $\displaystyle\int_{-\pi}^{-\gamma}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}d\theta=0.$

$$\int_{-\pi}^{-\gamma}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}d\theta=-\int_{\pi}^{\gamma}\frac{\sin(-a\phi)}{\sin(-\frac{1}{2}\phi)}d\phi=\int_{\gamma}^{\pi}\frac{\sin a\phi}{\sin\frac{1}{2}\phi}d\phi=0.$$

41. To prove that $\int_{-2\pi+\gamma}^{-\pi} \frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta = 0.$

$$\int_{-2\pi+\gamma}^{-\pi} \frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta = -\int_{2\pi-\gamma}^{\pi} \frac{\sin(-a\phi)}{\sin(-\frac{1}{2}\phi)}\,d\phi = \int_{\pi}^{2\pi-\gamma} \frac{\sin a\phi}{\sin\frac{1}{2}\phi}\,d\phi = 0.$$

42. In the next Article the value of a function of a, γ, δ, is considered in the particular case when $a\gamma = \infty$, $\gamma = 0$, $\frac{\gamma}{\delta} > 1$. Such a case would present itself were $\gamma = \pi a^{-\frac{1}{2}}$, $\delta = \pi a^{-1}$.

43. To prove that $\left.\begin{array}{l} a\gamma=\infty \\ \gamma=0 \\ \frac{\gamma}{\delta}>1 \end{array}\right| \int_{\delta}^{\gamma} \frac{\sin a\theta d\theta}{\sin\frac{1}{2}\theta} = 2\int_{a\delta}^{a\gamma} \frac{\sin\phi}{\phi}\,d\phi.$

Let m_1 be the integer $=$, or next $<$, $\frac{a\gamma}{\pi}$,

and let m_2 .... $>$, $\frac{a\delta}{\pi}$, then

$$\int_{\delta}^{\gamma} \frac{\sin a\theta d\theta}{\sin\frac{1}{2}\theta} = \left(\int_{\delta}^{\frac{m_2\pi}{a}} + \int_{\frac{m_2\pi}{a}}^{\frac{m_2+1}{a}\pi} + \ldots + \int_{\frac{m_1-1}{a}\pi}^{\frac{m_1\pi}{a}} + \int_{\frac{m_1\pi}{a}}^{\gamma}\right) \frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta$$

$$= \frac{1}{a}\left(\int_{a\delta}^{m_2\pi} + \int_{m_2\pi}^{(m_2+1)\pi} + \ldots + \int_{(m_1-1)\pi}^{m_1\pi} + \int_{m_1\pi}^{a\gamma}\right) \frac{\sin\phi}{\sin\frac{\phi}{2a}}\,d\phi.$$

These definite integrals alternate in sign.

Now, since $\sin\frac{\phi}{2a} < \frac{\phi}{2a}$, and $\frac{\sin\frac{\phi}{2a}}{1-\sin^2\frac{\phi}{2a}} > \frac{\phi}{2a}$;

$$\therefore \frac{1}{\sin\frac{\phi}{2a}} > \frac{2a}{\phi}, \text{ and } \frac{1}{\sin\frac{\phi}{2a}} < \frac{2a}{\phi} + \sin\frac{\phi}{2a}.$$

$$\therefore \int_\delta^\gamma \frac{\sin a\theta d\theta}{\sin\frac{1}{2}\theta} \text{ is in value intermediate to}$$

$$\frac{1}{a}\left(2a\int_{a\delta}^{a\gamma}\frac{\sin\phi}{\phi}\,d\phi\right)$$

$$+\frac{1}{a}\left(\int_{a\delta}^{m_2\pi}+\int_{(m_2+1)\pi}^{(m_2+2)\pi}+\ldots \text{ every odd term}\right)\sin\phi\sin\frac{\phi}{2a}\,d\phi,$$

and $\frac{1}{a}\left(2a\int_{a\delta}^{a\gamma}\frac{\sin\phi}{\phi}\,d\phi\right)$

$$+\frac{1}{a}\left(\int_{m_2\pi}^{(m_2+1)\pi}+\int_{(m_2+2)\pi}^{(m_2+3)\pi}+\ldots \text{ every even term}\right)\sin\phi\sin\frac{\phi}{2a}\,d\phi.$$

Now the two second terms of these expressions are nearer zero than

$$\frac{1}{a}\int_{a\delta}^{a\gamma} d\phi, \quad \text{or } (\gamma-\delta).$$

So it follows that $\Bigg|_{\substack{a\gamma=\infty\\ \gamma=0\\ \frac{\gamma}{\delta}>1}} \int_\delta^\gamma \frac{\sin a\theta . d\theta}{\sin\frac{1}{2}\theta} = 2\Bigg|^{a=\infty}\int_{a\delta}^{a\gamma}\frac{\sin\phi d\phi}{\phi}.$

44. To prove that $\Bigg|_{\substack{a\gamma=\infty\\ \gamma=0\\ \frac{\gamma}{\delta}>1}} \int_\delta^\gamma \frac{\cos a\theta d\theta}{\sin\frac{1}{2}\theta} = 2\int_{a\delta}^{a\gamma}\frac{\cos\phi d\phi}{\phi}.$

Let $(2n_1+1)$ be the odd integer =, or next >, $\frac{2a\delta}{\pi}$, and

let $(2n_2+1)$ .... <, $\frac{2a\gamma}{\pi}$, then

$$\int_\delta^\gamma\frac{\cos a\theta d\theta}{\sin\frac{1}{2}\theta}=\left(\int_\delta^{\frac{2n_1+1}{2a}\pi}+\int_{\frac{2n_1+1}{2a}\pi}^{\frac{2n_1+3}{2a}\pi}+\ldots+\int_{\frac{2n_2-1}{2a}\pi}^{\frac{2n_2+1}{2a}\pi}+\int_{\frac{2n_2+1}{2a}\pi}^{\gamma}\right)\frac{\cos a\theta d\theta}{\sin\frac{1}{2}\theta}$$

$$=\frac{1}{a}\left(\int_{a\delta}^{(2n_1+1)\frac{\pi}{2}}+\int_{\frac{2n_1+1}{2}\pi}^{\frac{2n_1+3}{2}\pi}+\ldots+\int_{\frac{2n_2-1}{2}\pi}^{\frac{2n_2+1}{2}\pi}+\int_{\frac{2n_2+1}{2}\pi}^{a\gamma}\right)\frac{\cos\phi d\phi}{\sin\frac{\phi}{2a}}.$$

These definite integrals alternate in sign.

As in Art. 43, $\int_\delta^\gamma \frac{\cos a\theta d\theta}{\sin\frac{1}{2}\theta}$ is in value intermediate to

$$\frac{1}{a}\left(2a\int_{a\delta}^{a\gamma}\frac{\cos\phi d\phi}{\phi}\right)$$

$$+\frac{1}{a}\left(\int_{a\delta}^{(2n_1+1)\frac{\pi}{2}}+\int_{\frac{2n_1+3}{2}\pi}^{\frac{2n_1+5}{2}\pi}+\dots\text{ every odd term}\right)\cos\phi\sin\frac{\phi}{2a}d\phi,$$

and $\frac{1}{a}\left(2a\int_{a\delta}^{a\gamma}\frac{\cos\phi d\phi}{\phi}\right)$

$$+\frac{1}{a}\left(\int_{(2n_1+1)\frac{\pi}{2}}^{\frac{2n_1+3}{2}\pi}+\int_{\frac{2n_1+5}{2}\pi}^{\frac{2n_1+7}{2}\pi}+\dots\text{ every even term}\right)\cos\phi\sin\frac{\phi}{2a}d\phi.$$

Now the two second terms of these expressions are nearer zero than

$$\frac{1}{a}\int_{a\delta}^{a\gamma}d\phi,\quad\text{or}\quad(\gamma-\delta).$$

So it follows that

$$\Big|_{\substack{a\gamma=\infty\\ \gamma=0\\ \frac{\gamma}{\delta}>1}}\int_\delta^\gamma\frac{\cos a\theta d\theta}{\sin\frac{1}{2}\theta}=2\int_{a\delta}^{a\gamma}\frac{\cos\phi d\phi}{\phi}.$$

45. It will be readily seen to follow from Arts. 37, 39, 40, 41, 43, that, provided -2π, γ_1, γ_2, $+2\pi$ are in order, and $(2\pi+\gamma_1)a$, $(2\pi-\gamma_2)a$, both infinite,

$$\Big|^{a=\infty}\int_{\gamma_1}^{\gamma_2}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}d\theta=2\int_{a\gamma_1}^{a\gamma_2}\frac{\sin\phi}{\phi}d\phi.$$

46. Supposing δ, γ, to have values as in Art. 42, then

$$\int_{2\pi-\gamma}^{2\pi-\delta} \frac{\sin a\theta}{\sin \frac{1}{2}\theta}\,d\theta = \int_{-\gamma}^{-\delta} \frac{\sin a\,(2\pi+\phi)}{\sin \frac{1}{2}(2\pi+\phi)}\,d\phi$$

$$= -\sin 2a\pi \int_{-\gamma}^{-\delta} \frac{\cos a\phi}{\sin \frac{\phi}{2}}\,d\phi - \cos 2a\pi \int_{-\gamma}^{-\delta} \frac{\sin a\phi}{\sin \frac{\phi}{2}}\,d\phi.$$

The value of this will be ultimately infinite, except when $\sin 2a\pi$ vanishes. But, when $2a$ is an integer, we may always consider $\sin 2a\pi \displaystyle\int_{-\gamma}^{-\delta} \frac{\cos a\phi}{\sin \frac{\phi}{2}}\,d\phi$ as vanishing, for

$$\sin\left(2a+\frac{c}{a}\right)\pi \int_{-\gamma}^{-\frac{c}{a}} \frac{d\phi}{\sin\frac{\phi}{2}} = 2\sin\left(2a+\frac{c}{a}\right)\pi.\left(\log\tan\frac{\gamma}{4} - \log\tan\frac{c}{4a}\right),$$

which vanishes, because $\pm \sin \frac{c}{a}\pi \log \sin \frac{c}{4a}$ ultimately vanishes.

So at this point it is necessary to introduce the restriction of $2a =$ an integer. In our applications $2a$ will always be an odd integer, so suppose $2a = 2s+1$, then

$$\left|^{s=\infty} \int_{2\pi-\gamma}^{2\pi-\delta} \frac{\sin \frac{2s+1}{2}\theta}{\sin \frac{1}{2}\theta}\,d\theta = -2\cos(2s+1)\pi \int_{-a\gamma}^{-a\delta} \frac{\sin\phi}{\phi}\,d\phi.\right.$$

Similarly

$$\left|^{s=\infty} \int_{-2\pi+\delta}^{-2\pi+\gamma} \frac{\sin \frac{2s+1}{2}\theta}{\sin \frac{1}{2}\theta}\,d\theta = -2\cos(2s+1)\pi \int_{a\delta}^{a\gamma} \frac{\sin\phi}{\phi}\,d\phi.\right.$$

47. We can now extend the conclusion arrived at in Art. 45. We have, provided $2a$ is an odd integer, and -2π, γ_1, γ_2, $+2\pi$, not out of order, $(2a=2s+1)$,

$$\Bigg|^{s=\infty}\int_{\gamma_1}^{\gamma_2}\frac{\sin\frac{2s+1}{2}\theta}{\sin\frac{1}{2}\theta}d\theta=2\left(\int_{a\gamma_1}^{a\gamma_2}+\int_{a(2\pi+\gamma_1)}^{a(2\pi+\gamma_2)}+\int_{a(-2\pi+\gamma_1)}^{a(-2\pi+\gamma_2)}\right)\frac{\sin\phi}{\phi}d\phi.$$

In this equation $\int_{a(2\pi+\gamma_1)}^{a(2\pi+\gamma_2)}$ vanishes, except when $a(2\pi+\gamma_1)$ is finite, and $\int_{a(-2\pi+\gamma_1)}^{a(-2\pi+\gamma_2)}$ except when $a(-2\pi+\gamma_2)$ is finite. So the equation immediately follows from the preceding Articles.

48. Since $2\cos\theta+2\cos 2\theta+\ldots+2\cos s\theta$

$$=\frac{2\cos\frac{s+1}{2}\theta\sin\frac{s\theta}{2}}{\sin\frac{1}{2}\theta}=\frac{-\sin\frac{1}{2}\theta+\sin\left(s+\frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta},$$

we have $$\int_0^{\pi}\frac{\sin\left(s+\frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta}d\theta=\pi+2\Bigg|_0^{\pi}\left(\sin\theta+\frac{\sin 2\theta}{2}+\ldots\right)=\pi.$$

Now $$2\Bigg|^{a=\infty}\int_0^{a\pi}\frac{\sin\phi d\phi}{\phi}=\Bigg|^{a=\infty}\int_0^{\pi}\frac{\sin\left(s+\frac{1}{2}\right)\theta d\theta}{\sin\frac{1}{2}\theta}.$$

We have therefore the well-known integral

$$\int_0^{\infty}\frac{\sin\phi d\phi}{\phi}=\frac{\pi}{2}.$$

49. Since $$\frac{\sin\psi}{\psi}=1-\frac{\psi^2}{\underline{|3}}+\frac{\psi^4}{\underline{|5}}-\dots,$$

$$\therefore \int_0^\psi \frac{\sin\psi d\psi}{\psi}=\psi-\frac{\psi^3}{3\underline{|3}}+\frac{\psi^5}{5\underline{|5}}-\dots,$$

and $$\int_\psi^\infty \frac{\sin\psi d\psi}{\psi}=\frac{\pi}{2}-\psi+\frac{\psi^3}{3\underline{|3}}-\frac{\psi^5}{5\underline{|5}}+\dots$$

50. Let c be a + quantity not infinitely great, then

$$\Big|^{s=\infty}\left(\sin\frac{c\pi}{2s}+\sin\frac{2c\pi}{2s}+\dots+\sin\frac{sc\pi}{2s}-\frac{4s}{\pi}\cdot\frac{1-\cos\frac{c\pi}{2}}{2c}\right)$$

$$=\Bigg|^{s=\infty}\left\{\frac{\cos\frac{c\pi}{4s}-\cos\frac{(2s+1)c\pi}{4s}}{2\sin\frac{c\pi}{4s}}-\frac{4s}{\pi}\cdot\frac{1-\cos\frac{c\pi}{2}}{2c}\right\}$$

$$=\Bigg|^{s=\infty}\frac{\frac{c\pi}{4s}\cos\frac{c\pi}{4s}-\frac{c\pi}{4s}\cos\frac{c\pi}{2}\cos\frac{c\pi}{4s}+\frac{c\pi}{4s}\sin\frac{c\pi}{2}\sin\frac{c\pi}{4s}-\sin\frac{c\pi}{4s}+\sin\frac{c\pi}{4s}\cos\frac{c\pi}{2}}{2\frac{c\pi}{4s}\sin\frac{c\pi}{4s}}$$

$$=\Bigg|^{s=\infty}\frac{\left(\frac{c\pi}{4s}\cos\frac{c\pi}{4s}-\sin\frac{c\pi}{4s}\right)\left(1-\cos\frac{c\pi}{2}\right)+\frac{c\pi}{4s}\sin\frac{c\pi}{2}\sin\frac{c\pi}{4s}}{2\frac{c\pi}{4s}\sin\frac{c\pi}{4s}}=\frac{1}{2}\sin\frac{c\pi}{2}.$$

So, dividing both sides by s, and integrating between finite limits 0, c,

$$\Big|^{s=\infty}\int_0^c\frac{1}{s}\left(\sin\frac{c\pi}{2s}+\sin\frac{2c\pi}{2s}+\dots+\sin\frac{sc\pi}{2s}\right)dc$$

$$=\frac{2}{\pi}\int_0^c\left\{\frac{c}{\underline{|2}}\left(\frac{\pi}{2}\right)^2-\frac{c^3}{\underline{|4}}\left(\frac{\pi}{2}\right)^4+\dots\right\}dc$$

$$=\frac{2}{\pi}\left\{\frac{1}{2\underline{|2}}\left(\frac{c\pi}{2}\right)^2-\frac{1}{4\underline{|4}}\left(\frac{c\pi}{4}\right)^4+\dots\right\};$$

i. e. $$\Big|^{s=\infty}\left\{\left(1-\cos\frac{c\pi}{2s}\right)+\left(\frac{1}{2}-\frac{1}{2}\cos\frac{2c\pi}{2s}\right)+\dots+\left(\frac{1}{s}-\frac{1}{s}\cos\frac{sc\pi}{2s}\right)\right\}$$
$$=\frac{1}{2\underline{|2}}\left(\frac{c\pi}{2}\right)^2-\frac{1}{4\underline{|4}}\left(\frac{c\pi}{4}\right)^4+\dots$$

If we put $\theta=\frac{c\pi}{2}$, we have the equation

$$\Big|^{s=\infty}\left\{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{s}-\left(\cos\frac{\theta}{s}+\cos\frac{2\theta}{s}+\dots+\cos\frac{s\theta}{s}\right)\right\}$$
$$=\frac{\theta^2}{2\underline{|2}}-\frac{\theta^4}{4\underline{|4}}+\frac{\theta^6}{6\underline{|6}}-\dots$$

51. Since $2\sin\theta+2\sin2\theta+\dots+2\sin s\theta$

$$=\frac{2\sin\frac{s+1}{2}\theta\sin\frac{s\theta}{2}}{\sin\frac{1}{2}\theta}=\frac{\cos\frac{1}{2}\theta-\cos\left(s+\frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta},$$

$$\therefore\int_\delta^\pi\frac{\cos\left(s+\frac{1}{2}\right)\theta d\theta}{\sin\frac{1}{2}\theta}=\int_\delta^\pi\cot\frac{1}{2}\theta d\theta+2\Big|_\delta^\pi\left(\cos\theta+\frac{\cos2\theta}{2}+\dots+\frac{\cos s\theta}{s}\right)$$

$$=2\log\frac{\sin\frac{\pi}{2}}{\sin\frac{\delta}{2}}+2\left\{-1+\frac{1}{2}-\frac{1}{3}+\dots+\frac{(-1)^s}{s}\right\}$$
$$-2\left(\cos\delta+\frac{\cos2\delta}{2}+\dots+\frac{\cos s\delta}{s}\right)$$

$$=-2\left\{1-\frac{1}{2}+\frac{1}{3}+\dots-\frac{(-1)^s}{s}-\log2\right\}-2\log\frac{2\sin\frac{\delta}{2}}{\delta}$$
$$-2\log s\delta+2\left(\log s-1-\frac{1}{2}-\frac{1}{3}-\dots-\frac{1}{s}\right)$$
$$+2\left\{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{s}-\left(\cos\delta+\cos2\delta+\dots+\cos\frac{s\delta}{s}\right)\right\}.$$

Now let s have a value indefinitely great, and $s\delta$ be finite, so that δ is indefinitely small, this gives (38, 44)

$$\int_{s\delta}^{s\pi} \frac{\cos\phi d\phi}{\phi} = -C - \log s\delta + \frac{(s\delta)^2}{2\lfloor\underline{2}} - \frac{(s\delta)^4}{4\lfloor\underline{4}} + \dots$$

where $$C = \Big|^{s=\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{s} - \log s\right).$$

Since $$C = \frac{1}{2\lfloor\underline{2}} - \frac{1}{4\lfloor\underline{4}} + \dots - \int_1^\infty \frac{\cos\phi}{\phi} d\phi,$$

its value is finite. It has been computed, and shewn that

$$C = 0{\cdot}5772156\dots\dots$$

52. Besides the expansions of (49) and (51) the following, which can be readily established, are worthy of notice.

$$\int_0^x \frac{\sin\phi d\phi}{\phi} + \int_0^{\frac{1}{x}} \frac{\sin\phi d\phi}{\phi} = 2\int_0^1 \frac{\sin\phi d\phi}{\phi} + \frac{2(\log x)^2 \cos 1}{\lfloor\underline{2}}$$
$$+ \frac{2(\log x)^4}{\lfloor\underline{4}} \Big|^{z=1} \left(z\frac{d}{dz}\right)^3 \sin z + \dots + \frac{2(\log x)^{2n}}{\lfloor\underline{2n}} \Big|^{z=1} \left(z\frac{d}{dz}\right)^{2n-1} \sin z + \dots$$

$$\int_0^x \frac{\sin\phi d\phi}{\phi} - \int_0^{\frac{1}{x}} \frac{\sin\phi d\phi}{\phi} = 2\log x \sin 1$$
$$+ \frac{2(\log x)^3}{\lfloor\underline{3}} \Big|^{z=1} \left(z\frac{d}{dz}\right)^2 \sin z + \dots$$
$$+ \frac{2(\log x)^{2n+1}}{\lfloor\underline{2n+1}} \Big|^{z=1} \left(z\frac{d}{dz}\right)^{2n} \sin z + \dots$$

$$\int_x^\infty \frac{\cos\phi d\phi}{\phi} + \int_{\frac{1}{x}}^\infty \frac{\cos\phi d\phi}{\phi} = 2\int_1^\infty \frac{\cos\phi d\phi}{\phi} + \frac{2(\log x)^2}{\lfloor\underline{z}} \sin 1$$
$$- \frac{2(\log x)^4}{\lfloor\underline{4}} \Big|^{z=1} \left(z\frac{d}{dz}\right)^3 \cos z - \dots$$
$$- \frac{2(\log x)^{2n}}{\lfloor\underline{2n}} \Big|^{z=1} \left(z\frac{d}{dz}\right)^{2n-1} \cos z - \dots$$

$$\int_x^\infty \frac{\cos\phi d\phi}{\phi} - \int_{\frac{1}{x}}^\infty \frac{\cos\phi d\phi}{\phi} = -2\log x \cos 1$$

$$-\frac{2(\log x)^3}{3\underline{|3}}\Big|^{z=1}\left(z\frac{d}{dz}\right)^2 \cos z - \dots$$

$$-\frac{2(\log x)^{2n+1}}{\underline{|2n+1}}\Big|^{z=1}\left(z\frac{d}{dz}\right)^{2n} \cos z - \dots$$

$$\left(z\frac{d}{dz}\right)^r \sin z = \Delta 0^r z \cos z - \frac{\Delta^2 0^r}{\underline{|2}} z^2 \sin z - \frac{\Delta^3 0^r}{\underline{|3}} z^3 \cos z + \dots$$

$$+\frac{\Delta^{4s-3}0^r}{\underline{|4s-3}} z^{4s-3} \cos z - \frac{\Delta^{4s-2}0^r}{\underline{|4s-2}} z^{4s-2} \sin z - \frac{\Delta^{4s-1}0^r}{\underline{|4s-1}} z^{4s-1} \cos z$$

$$+\frac{\Delta^{4s}0^r}{\underline{|4s}} z^{4s} \sin z + \dots$$

$$\left(z\frac{d}{dz}\right)^r \cos z = -\Delta 0^r z \sin z - \frac{\Delta^2 0^r}{\underline{|2}} z^2 \cos z + \frac{\Delta^3 0^r}{\underline{|3}} z^3 \sin z + \dots$$

$$-\frac{\Delta^{4s-3}0^r}{\underline{|4s-3}} z^{4s-3} \sin z - \frac{\Delta^{4s-2}0^r}{\underline{|4s-2}} z^{4s-2} \cos z + \frac{\Delta^{4s-1}0^r}{\underline{|4s-1}} z^{4s-1} \sin z$$

$$+\frac{\Delta^{4s}0^r}{\underline{|4s}} z^{4s} \cos z - \dots$$

53. Suppose that $f(x)$ is a function having no true discontinuity as x ranges between certain known values. It may happen that $f'(x)$ becomes infinite for some value of x between these values. Let $y = f(x)$ be the equation to a curve having that property, x_1, x_2, being the values between which x ranges, and x' the value that makes $f'(x)$ infinite. A curve can be drawn, as $APP'Q$ (fig. 3), such that, for every value of x, its tangent is parallel to the tangent of the curve APQ. If $\eta = f(x) + f_1(x)$ be such a curve, $f_1(x)$ will be zero until $x = x'$, and $f_1'(x)$ will be zero, except when $x = x'$, when it will be infinite, for $\frac{d(y-\eta)}{dx}$ is infinite when $x = x'$. Hence $f'(x)$ and $\{f'(x) + f_1'(x)\}$ will always confound with each other.

In such cases by the definite integral $\int_{x_1}^{x_2} f'(x)\,dx$ we always mean $f(x_2)-f(x_1)$, although $f'(x)$ becomes infinite between the limits. In this case $f'(x)$ must never be confounded with another function always equal to it when finite; our knowledge of it must be considered to spring not from experience, but from the mental concept of the law expressed.

54. Suppose $F(\theta)$, $f(\theta)$, to be such functions of θ that for some integral value of x,

$$\int_{\gamma_1}^{\theta} F^{(n+1)}(\theta)\left\{\left(\int_{\gamma_1}^{\theta}\right)^{n+1} f(\theta)\,d\theta^{n+1}\right\} d\theta = 0,$$

then shall

$$\int_{\gamma_1}^{\theta} F(\theta)f(\theta)\,d\theta = F(\theta)\int_{\gamma_1}^{\theta} f(\theta)\,d\theta - F'(\theta)\left(\int_{\gamma_1}^{\theta}\right)^{2} f(\theta)\,d\theta^{2} + \dots$$
$$+ (-1)^n F^{(n)}(\theta)\left(\int_{\gamma_1}^{\theta}\right)^{n+1} f(\theta)\,d\theta^{n+1}.$$

For since $F(\theta)f(\theta) + F'(\theta)\int_{\gamma_1}^{\theta} f(\theta)\,d\theta \equiv \dfrac{d}{d\theta}\left\{F(\theta)\int_{\gamma_1}^{\theta} f(\theta)\,d\theta\right\}$,

$$\therefore \int_{\gamma_1}^{\theta} F(\theta)f(\theta)\,d\theta + \int_{\gamma_1}^{\theta}\left\{F'(\theta)\int_{\gamma_1}^{\theta} f(\theta)\,d\theta\right\} d\theta \equiv F(\theta)\int_{\gamma_1}^{\theta} f(\theta)\,d\theta,$$

and similarly

$$-\int_{\gamma_1}^{\theta} F'(\theta)\left\{\int_{\gamma_1}^{\theta} f(\theta)\,d\theta\right\} d\theta - \int_{\gamma_1}^{\theta} F''(\theta)\left\{\left(\int_{\gamma_1}^{\theta}\right)^{2} f(\theta)\,d\theta^{2}\right\} d\theta$$
$$\equiv -F'(\theta)\left(\int_{\gamma_1}^{\theta}\right)^{2} f(\theta)\,d\theta$$

$$\dots\dots\dots\dots\dots\dots \equiv \dots\dots\dots\dots\dots\dots$$

$$(-1)^n\int_{\gamma_1}^{\theta} F^{(n)}(\theta)\left\{\left(\int_{\gamma_1}^{\theta}\right)^{n} f(\theta)\,d\theta^{n}\right\}d\theta + (-1)^n\int_{\gamma_1}^{\theta} F^{(n+1)}(\theta)\left\{\left(\int_{\gamma_1}^{\theta}\right)^{n+1} f(\theta)\,d\theta^{n+1}\right\}d\theta$$
$$\equiv (-1)^n F^{(n)}(\theta)\left(\int_{\gamma_1}^{\theta}\right)^{n+1} f(\theta)\,d\theta^{n+1}.$$

Whence, by addition, the proposition.

55. In like manner, from the identity

$$\int_\theta^\omega F(\theta) f(\theta)\, d\theta - \int_\theta^\omega F^{(n+1)}(\theta) \left\{ \left(\int_\theta^\omega\right)^{n+1} f(\theta)\, d\theta^{n+1} \right\} d\theta$$

$$\equiv F(\theta) \int_\theta^\omega f(\theta)\, d\theta + F'(\theta) \left(\int_\theta^\omega\right)^2 f(\theta)\, d\theta^2 + \ldots + F^{(n)}(\theta) \left(\int_\theta^\omega\right)^{n+1} f(\theta)\, d\theta^{n+1},$$

we obtain

$$\int_\theta^\omega F(\theta) f(\theta)\, d\theta = F(\theta) \int_\theta^\omega f(\theta)\, d\theta + F'(\theta) \left(\int_\theta^\omega\right)^2 f(\theta)\, d\theta^2 + \ldots$$

$$+ F^{(n)}(\theta) \left(\int_\theta^\omega\right)^{n+1} f(\theta)\, d\theta^{n+1},$$

provided only that

$$\int_\theta^\omega F^{(n+1)}(\theta) \left\{ \left(\int_\theta^\omega\right)^{n+1} f(\theta)\, d\theta^{n+1} \right\} d\theta = 0.$$

(54) applies when $\theta > \gamma_1$, and (55) when $\theta < \omega$. In these formulæ $\int_{\gamma_1}^\theta f(\theta)\, d\theta$, $\int_\theta^\omega f(\theta)\, d\theta$, must be finite, which will secure that the successive integrals are finite. There must be no true discontinuity in any of the functions $F^{(r)}(\theta)$, a fact always implied in functions whose differential coefficients are taken (vide Art. 14), but there may be any amount of false discontinuity.

56. In a subsequent part of this Essay we shall frequently employ differential coefficients of negative orders, as $\frac{d^{-r}y}{dx^{-r}}$. Now it is manifest that the symbol $\frac{d^{-r}y}{dx^{-r}}$ should obey the law

$$\frac{d^{+r}}{dx^{+r}} \left(\frac{d^{-r}y}{dx^{-r}} \right) = y;$$

also it is evident that

$$\frac{d^{+r}}{dx^{+r}} \left(\int_{\gamma_1}^x \int_{\gamma_2}^x \cdots \int_{\gamma_r}^x y\, dx^r \right) = \frac{d^r}{dx^r} \left(\int_\gamma^x \right)^r y\, dx^r$$

$$= \frac{d^r}{dx^r} \left\{ \left(\int_\gamma^x \right)^r y\, dx^r + C_0 + C_1 x + \ldots + C_{r-1} x^{r-1} \right\}.$$

In this $C_0, C_1, \dots C_{r-1}$ are determinate functions of $\gamma, \gamma_1, \gamma_2, \dots \gamma_r$, if

$$\int_{\gamma_1}^{x}\int_{\gamma_2}^{x}\dots\int_{\gamma_r}^{x} y\,dx^r \equiv \left(\int_{\gamma}^{x}\right)^r y\,dx^r + C_0 + C_1 x + \dots + C_{r-1}x^{r-1}.$$

In fáct, r' being less than r,

$$\int_{\gamma_{r'}}^{x}\int_{\gamma_{r'+1}}^{x}\dots\int_{\gamma_r}^{x} y\,dx^{r-r'+1} = \left(\int_{\gamma}^{x}\right)^{r-r'+1} y\,dx^{r-r'+1} + \underline{\lfloor r'-1}\, C_{r-1}$$
$$+ \frac{\underline{\lfloor r'}}{1} C_{r}x + \dots + \frac{\underline{\lfloor r-1}}{\underline{\lfloor r-r'}} C_{r-1}x^{r-r'}.$$

So that

$$\int_{\gamma_{r'}}^{0}\int_{\gamma_{r'+1}}^{x}\dots\int_{\gamma_r}^{x} y\,dx^{r-r'+1} = \int_{\gamma}^{0}\left(\int_{\gamma}^{x}\right)^{r-r'} y\,dx^{r-r'}\,dx + \underline{\lfloor r'-1}\, C_{r'-1}.$$

We shall attach to $\dfrac{d^{-r}y}{dx^{-r}}$ the most general signification that it can bear. So that

$$\frac{d^{-r}y}{dx^{-r}} \equiv \int_{\gamma_1}^{x}\int_{\gamma_2}^{x}\dots\int_{\gamma_r}^{x} y\,dx^r \equiv \left(\int_{\gamma}^{x}\right)^r y\,dx^r + C_0 + C_1 x + \dots + C_{r-1}x^{r-1}.$$

Observe that $\int_{\gamma_1}^{x} = -\int_{x}^{\gamma_1}$, and the integrations only exceptionally extend over infinite values.

57. If $f(\theta)$ is finite for values of θ between γ_1 and the greatest value θ is supposed to have,

$$\underline{\lfloor n}\left(\int_{\gamma_1}^{\theta}\right)^{n+1} f(\theta)\,d\theta^{n+1} \equiv \theta^n\int_{\gamma_1}^{\theta} f(\theta)\,d\theta - \frac{n}{1}\,.\,\theta^{n-1}\int_{\gamma_1}^{\theta}\theta f(\theta)\,d\theta$$
$$+\frac{n(n-1)}{1\,.\,2}\theta^{n-2}\int_{\gamma_1}^{\theta}\theta^2 f(\theta)\,d\theta - \dots + (-1)^{n-1}\frac{n}{1}\,.\,\theta\int_{\gamma_1}^{\theta}\theta^{n-1}f(\theta)\,d\theta$$
$$+(-1)^n\int_{\gamma_1}^{\theta}\theta^n f(\theta)\,d\theta.$$

Both sides vanish when $\theta = \gamma_1$, and if we differentiate with respect to θ, we get

$$n\left\{\underline{\lfloor n-1}\left(\int_{\gamma_1}^{\theta}\right)^n f(\theta)\,d\theta^n\right\} \equiv n\left\{\theta^{n-1}\int_{\gamma_1}^{\theta} f(\theta)\,d\theta - \frac{n-1}{1}\,.\,\theta^{n-2}\int_{\gamma_1}^{\theta}\theta f(\theta)\,d\theta + \dots\right.$$
$$\left.+(-1)^{n-1}\int_{\gamma_1}^{\theta}\theta^{n-1}f(\theta)\,d\theta\right\} + \left\{1 - \frac{n}{1} + \frac{n(n-1)}{1\,.\,2} + \dots + (-1)^{n-1}\frac{n}{1} + (-1)^n\right\}\theta^n f(\theta),$$

and the formula is true when $n=1$, for

$$\left(\int_{\gamma_1}^{\theta}\right)^2 f(\theta)\,d\theta^2 \equiv \theta\int_{\gamma_1}^{\theta} f(\theta)\,d\theta - \int_{\gamma_1}^{\theta}\theta f(\theta)\,d\theta.$$

Therefore the formula is true generally.

58. In like manner, if $f(\theta)$ is finite for values of θ between ω and the least value θ is supposed to have,

$$(-1)^n \lfloor\underline{n}\left(\int_{\theta}^{\omega}\right)^{n+1} f(\theta)\,d\theta^{n+1} \equiv \theta^n\int_{\theta}^{\omega} f(\theta)\,d\theta - \frac{n}{1}\theta^{n-1}\int_{\theta}^{\omega}\theta f(\theta)\,d\theta$$

$$+\frac{n(n-1)}{1\,.\,2}\theta^{n-2}\int_{\theta}^{\omega}\theta^2 f(\theta)\,d\theta + \ldots + (-1)^{n-1}\frac{n}{1}\theta\int_{\theta}^{\omega}\theta^{n-1} f(\theta)\,d\theta$$

$$+(-1)^n\int_{\theta}^{\omega}\theta^n f(\theta)\,d\theta.$$

Both sides vanish when $\theta=\omega$, and, if we differentiate with respect to θ, we get

$$n\left\{(-1)^{n-1}\lfloor\underline{n-1}\left(\int_{\theta}^{\omega}\right)^n f(\theta)\,d\theta^n\right\}$$

$$\equiv n\left\{\theta^{n-1}\int_{\theta}^{\omega} f(\theta)\,d\theta - \frac{n-1}{1}\theta^{n-2}\int_{\theta}^{\omega}\theta f(\theta)\,d\theta + \ldots + (-1)^{n-1}\int_{\theta}^{\omega}\theta^{n-1} f(\theta)\,d\theta\right\}$$

$$-\left\{1-n+\frac{n(n-1)}{1\,.\,2}+\ldots+(-1)^{n-1}n+(-1)^n\right\}\theta^n f(\theta),$$

and the formula is true when $n=1$, for

$$-\left(\int_{\theta}^{\omega}\right)^2 f(\theta)\,d\theta^2 \equiv \theta\int_{\theta}^{\omega} f(\theta)\,d\theta - \int_{\theta}^{\omega}\theta f(\theta)\,d\theta.$$

Therefore the formula is true generally.

59. In the next Article the symbol $\int_{\gamma_1}^{\gamma_2}\sqrt{f'(\theta)^2}\,d\theta$ will occur. The meaning of it requires no explanation, except when $f'(\theta)$ becomes infinite between the limits, $f(\theta)$ remaining always finite and not truly discontinuous.

Trace the curve $y=f(\theta)$, (fig. 4), and suppose that, at every point where $f'(\theta)$ changes sign, the curve revolves round the parallel through it to the axis of θ through two right angles, and let this process be repeated till a curve is obtained having throughout $\frac{dy}{d\theta} = +\sqrt{f'(\theta)^2}$. The extreme ordinates of this curve will have $\int_{\gamma_1}^{\gamma_2}\sqrt{f'(\theta)^2}\,d\theta$ for their difference, and this expression will always be finite, if the number of times that $f'(\theta)$ changes its sign is finite. And, the mental concept of $\int_{\gamma_1}^{\gamma_2}\sqrt{f'(\theta)^2}\,d\theta$ being definite, its value is definite, although $f'(\theta)$ become infinite between the limits.

60. We can now establish the following proposition. Suppose $f(\theta)$ to be a function of θ such that, for all values of θ between -2π and $+2\pi$, $\int_{-2\pi}^{\theta}\sqrt{f'(\theta)^2}\,d\theta$ is finite, and that $\int_{-\delta}^{+\delta}\sqrt{f'(\theta)^2}\,d\theta$ can be made less than any assignable quantity by taking δ small enough, even though $a\delta$ be infinite. Then, -2π, γ_1, γ_2, $+2\pi$, not being out of order, and $a(2\pi+\gamma_1)$, $a(2\pi-\gamma_2)$, being infinite, shall

$$\Big|^{a=\infty}\int_{\gamma_1}^{\gamma_2} f(\theta)\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta = f(0)\int_{\gamma_1}^{\gamma_2}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta.$$

We have identically,

$$\int_{\gamma_1}^{\gamma_2} f(\theta)\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta - f(0)\int_{\gamma_1}^{\gamma_2}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta = \{f(\gamma_2)-f(0)\}\int_{\gamma_1}^{\gamma_2}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta$$

$$-\int_{\gamma_1}^{\gamma_2} f'(\theta)\int_{\gamma_1}^{\theta}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta d\theta.$$

Firstly, suppose γ_2 negative, and $a\gamma_2$ infinite, then

$$\int_{\gamma_1}^{\gamma_2}\sqrt{f'(\theta)^2\left(\int_{\gamma_1}^{\theta}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta\right)^2}\,d\theta$$

cannot be less than $\pm\int_{\gamma_1}^{\gamma_2} f'(\theta)\int_{\gamma_1}^{\theta}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta\,d\theta$. The greatest value $\left(\int_{\gamma_1}^{\theta}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta\right)^2$ can have is indefinitely small, and $\int_{\gamma_1}^{\gamma_2}\sqrt{f'(\theta)^2}\,d\theta$ is, by hypothesis, finite,

$$\therefore \int_{\gamma_1}^{\gamma_2} f'(\theta)\int_{\gamma_1}^{\theta}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta\,d\theta \text{ vanishes.}$$

Also $\int_{\gamma_1}^{\gamma_2}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta$ vanishes, and $\{f(\gamma_2)-f(0)\}$, which cannot in value exceed $\int_{\gamma_1}^{\theta}\sqrt{f'(\theta)^2}\,d\theta$, is finite. So that in this case the proposition is proved.

Secondly, suppose $a\gamma_2$ indefinitely small, and $a\gamma_1$ infinite. Let γ' be negative, indefinitely small, and $a\gamma'$ infinite. Then, as in the first case,

$$\int_{\gamma_1}^{\gamma'} f(\theta)\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta - f(0)\int_{\gamma_1}^{\gamma'}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta = 0;$$

and, since

$$\int_{\gamma'}^{\gamma_2} f(\theta)\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta - f(0)\int_{\gamma'}^{\gamma_2}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta$$

$$=\{f(\gamma_2)-f(0)\}\int_{\gamma'}^{\gamma_2}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta - \int_{\gamma'}^{\gamma_2} f'(\theta)\int_{\gamma'}^{\theta}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta\,d\theta,$$

and

$$\pm\int_{\gamma'}^{\gamma_2} f'(\theta)\int_{\gamma'}^{\theta}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta\,d\theta \not> \int_{\gamma'}^{\gamma_2}\sqrt{f'(\theta)^2\left(\int_{\gamma'}^{\theta}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta\right)^2}\,d\theta$$

$$\not> 2\pi\int_{\gamma'}^{\gamma_2}\sqrt{f'(\theta)^2}\,d\theta,$$

it vanishes, for $\int_{\gamma'}^{\gamma_2}\sqrt{f'(\theta)^2}\,d\theta$, by hypothesis, vanishes. And, in this case, $f(\gamma_2)-f(0)$ vanishes. So that in this case the proposition is proved.

Thirdly, suppose both $a\gamma_2$, $a\gamma_1$, indefinitely small. Then, by the second case,

$$\int_{\gamma'}^{\gamma_2} f(\theta)\,\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta = f(0)\int_{\gamma'}^{\gamma_2}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta,$$

$$\int_{\gamma'}^{\gamma_1} f(\theta)\,\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta = f(0)\int_{\gamma'}^{\gamma_1}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta.$$

and, by subtraction, the proposition.

Fourthly, suppose γ_2 +, and $a\gamma_2$ infinite. Then, since

$$\int_{\gamma_1}^{\gamma_2} f(\theta)\,\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta = \int_{-\gamma_2}^{-\gamma_1} f(-\theta)\,\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta,$$

$$f(0)\int_{\gamma_1}^{\gamma_2}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta = f(0)\int_{-\gamma_2}^{-\gamma_1}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta,$$

the proposition follows at once, from the preceding cases, if γ_1 is +; and, by help of the identity $\int_{\gamma_1}^{0}+\int_{0}^{\gamma_2}=\int_{\gamma_1}^{\gamma_2}$, if γ_1 is −.

61. Supposing everything else as in Art. 60, and that $a(2\pi+\gamma_1)$ is finite, we have, if $a(2\pi+\gamma')$ is infinite, $(2\pi+\gamma')$ being indefinitely small,

$$\int_{\gamma_1}^{\gamma'} f(\theta)\,\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta = \int_{\gamma_1+2\pi}^{\gamma'+2\pi} f(\theta-2\pi)\,\frac{\sin a(\theta-2\pi)}{\sin\frac{1}{2}(\theta-2\pi)}\,d\theta$$

$$= \int_{\gamma_1+2\pi}^{\gamma'+2\pi} f(\theta-2\pi)\,\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta,$$

introducing the restriction $\cos 2a\pi = -1$ (vide Art. 46).

So we have

$$\int_{\gamma_1}^{\gamma} f(\theta)\,\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta = f(-2\pi)\int_{\gamma_1+2\pi}^{\gamma+2\pi}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta.$$

A similar method applies when $a\,(2\pi-\gamma_2)$ is finite.

We see from this, and Art. 47, that we can always write

$$\int_{\gamma_1}^{\gamma_2} f(\theta)\,\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta = 2\left\{f(-2\pi)\int_{a(2\pi+\gamma_1)}^{a(2\pi+\gamma_2)} + f(0)\int_{a\gamma_1}^{a\gamma_2}\right.$$

$$\left. + f(2\pi)\int_{a(-2\pi+\gamma_1)}^{a(-2\pi+\gamma_2)}\right\}\frac{\sin\phi}{\phi}\,d\phi,$$

when the conditions mentioned at commencement of Art. 60 are satisfied by $f(\theta)$.

62. It follows, from Art. 60, that if $f_1(\theta)$, $f_2(\theta)$, $\ldots f_n(\theta)$, be all functions such as there described, and finite in number,

$$\int_{\gamma_1}^{\theta_1} f_1(\theta)\,\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta = f_1(0)\int_{\gamma_1}^{\theta_1}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta$$

$$\int_{\theta_1}^{\theta_2} f_2(\theta)\,\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta = f_2(0)\int_{\theta_1}^{\theta_2}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta$$

$$\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\int_{\theta_{r-1}}^{0} f_r(\theta)\,\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta = f_r(0)\int_{\theta_{r-1}}^{0}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta$$

$$\int_{0}^{\theta_{r+1}} f_{r+1}(\theta)\,\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta = f_{r+1}(0)\int_{0}^{\theta_{r+1}}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta$$

$$\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\int_{\theta_{n-1}}^{\gamma_2} f_n(\theta)\,\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta = f_n(0)\int_{\theta_{n-1}}^{\gamma_2}\frac{\sin a\theta}{\sin\frac{1}{2}\theta}\,d\theta.$$

Now if $a\theta_{r-1}$, $a\theta_{r+1}$, are both infinite, addition gives us

$$\left\{\int_{\gamma_1}^{\theta_1} f_1(\theta) + \int_{\theta_1}^{\theta_2} f_2(\theta) + \dots + \int_{\theta_{r-1}}^{0} f_r(\theta) + \int_{0}^{\theta_{r+1}} f_{r+1}(\theta) + \dots \right.$$

$$\left. + \int_{\theta_{r-1}}^{\gamma_2} f_n(\theta)\right\} \frac{\sin a\theta}{\sin \frac{1}{2}\theta}\, d\theta = \pi \left\{f_r(0) + f_{r+1}(0)\right\},$$

and there is no difficulty in seeing what the result is in case $a\theta_{r-1}$, or $a\theta_{r+1}$, are finite. This shews in what manner the result of Art. 60 can be applied in the case of $f(\theta)$ being discontinuous. There is no difficulty in seeing what formulæ must be used in the case of Art. 61, in which $a(2\pi + \gamma_1)$, or $a(2\pi - \gamma_2)$, are not infinite.

63. The condition $\int_{-2\pi}^{\theta} \sqrt{f'(\theta)^2}\, d\theta$ not infinite excludes the application of the formulæ to a function such as $\sin a\theta$, (a infinite,) and the condition $\int_{-\delta}^{+\delta} \sqrt{f'(\theta)^2}\, d\theta$ evanescent with δ excludes such functions as $\sin \frac{1}{x}$, and $\frac{ax}{b}$, when $a\delta$ is not evanescent with δ.

64. Let $\gamma_2 + \beta = l_2$, $\gamma_1 + \beta = l_1$, and restrict β to range from $-\pi$ to $+\pi$, and suppose $-\pi$, l_1, l_2, $+\pi$ are not out of order; then neither -2π, $l_1 + \beta$, $l_2 + \beta$, $+2\pi$, nor -2π, $l_1 - \beta$, $l_2 - \beta$, $+2\pi$, are out of order, and

$$\Bigg|^{a=\infty} \int_{l_1}^{l_2} \frac{f(\theta) \sin a(\theta - \beta)\, d\theta}{\sin \frac{1}{2}(\theta - \beta)} = 2\left\{f(2\pi + \beta) \int_{a(-2\pi + l_1 - \beta)}^{a(-2\pi + l_2 - \beta)} \right.$$

$$\left. + f(\beta) \int_{a(l_1 - \beta)}^{a(l_2 - \beta)} + f(-2\pi + \beta) \int_{a(2\pi + l_1 - \beta)}^{a(2\pi + l_2 - \beta)} \right\} \frac{\sin \phi}{\phi}\, d\phi.$$

Similarly

$$\Bigg|^{a=\infty} \int_{l_1}^{l_2} \frac{f(\theta) \sin a(\theta + \beta)\, d\theta}{\sin \frac{1}{2}(\theta + \beta)} = 2\left\{f(2\pi - \beta) \int_{a(-2\pi + l_1 + \beta)}^{a(-2\pi + l_2 + \beta)} \right.$$

$$\left. + f(-\beta) \int_{a(l_1 + \beta)}^{a(l_2 + \beta)} + f(-2\pi - \beta) \int_{a(2\pi + l_1 + \beta)}^{a(2\pi + l_2 + \beta)} \right\} \frac{\sin \phi}{\phi}\, d\phi.$$

In these equations $2a$ must be an odd integer, whenever the terms involving $f(\pm 2\pi \pm \beta)$ do not vanish. Now, if $2a = 2s + 1$,

$$\frac{\sin a(\theta-\beta)}{\sin\frac{1}{2}(\theta-\beta)} = 1 + 2\sum_{r=1}^{r=s} \cos r(\theta-\beta),$$

and $$\frac{1}{2}\frac{\sin a(\theta-\beta)}{\sin\frac{1}{2}(\theta-\beta)} + \frac{1}{2}\frac{\sin a(\theta+\beta)}{\sin\frac{1}{2}(\theta+\beta)} = 1 + 2\sum_{r=1}^{r=s} \cos r\theta \cos r\beta,$$

$$\frac{1}{2}\frac{\sin a(\theta-\beta)}{\sin\frac{1}{2}(\theta-\beta)} - \frac{1}{2}\frac{\sin a(\theta+\beta)}{\sin\frac{1}{2}(\theta+\beta)} = 2\sum_{r=1}^{r=s} \sin r\theta \sin r\beta.$$

So that

$$\Big|^{a=\infty} \int_{l_1}^{l_2} f(\theta) \left\{\frac{1}{2} + \sum_{r=1}^{r=s} \cos r\theta \cos r\beta\right\} d\theta = \frac{1}{2}\Big\{ f(2\pi+\beta) \int_{a(-2\pi+l_1-\beta)}^{a(-2\pi+l_2-\beta)}$$
$$+ f(2\pi-\beta) \int_{a(-2\pi+l_1+\beta)}^{a(-2\pi+l_2+\beta)} + f(\beta) \int_{a(l_1-\beta)}^{a(l_2-\beta)} + (f-\beta) \int_{a(l_1+\beta)}^{a(l_2+\beta)}$$
$$+ f(-2\pi+\beta) \int_{a(2\pi+l_1-\beta)}^{a(2\pi+l_2-\beta)} + f(-2\pi-\beta) \int_{a(2\pi+l_1+\beta)}^{a(2\pi+l_2+\beta)}\Big\} \frac{\sin\phi}{\phi} d\phi,$$

and

$$\Big|^{a=\infty} \int_{l_1}^{l_2} f(\theta) \sum_{r=1}^{r=s} \sin r\theta \sin r\beta d\theta = \frac{1}{2}\Big\{ f(2\pi+\beta) \int_{a(-2\pi+l_1-\beta)}^{a(-2\pi+l_2-\beta)}$$
$$- f(2\pi-\beta) \int_{a(-2\pi+l_1+\beta)}^{a(-2\pi+l_2+\beta)} + f(\beta) \int_{a(l_1-\beta)}^{a(l_2-\beta)} - f(-\beta) \int_{a(l_1+\beta)}^{a(l_2+\beta)}$$
$$+ f(-2\pi+\beta) \int_{a(2\pi+l_1-\beta)}^{a(2\pi+l_2-\beta)} - f(-2\pi-\beta) \int_{a(2\pi+l_1+\beta)}^{a(2\pi+l_2+\beta)}\Big\} \frac{\sin\phi}{\phi} d\phi.$$

In these equations $\int_{a(-2\pi+l_1-\beta)}^{a(-2\pi+l_2-\beta)}$ vanishes save when $\pi - l_2$, $\pi + \beta$, both vanish, and similarly for the other like integrals: also $\int_{a(l_1-\beta)}^{a(l_2-\beta)}$ vanishes if either $(\beta - l_2)$ or $(l_1 - \beta)$ is + and finite, and $\int_{a(l_1+\beta)}^{a(l_2+\beta)}$ if either $(-\beta - l_2)$ or $(l_1 + \beta)$ is + and finite.

Now if $\Phi(\theta) = f_0(\theta)$ from $\theta = -\pi$ to $\theta = \lambda_1$

and

$$\begin{aligned} &= f_1(\theta) \;\ldots\ldots\; \theta = \lambda_1 \;\ldots\; \theta = \lambda_2 \\ &= f_2(\theta) \;\ldots\ldots\; \theta = \lambda_2 \;\ldots\; \theta = \lambda_3 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= f_n(\theta) \;\ldots\ldots\; \theta = \lambda_n \;\ldots\; \theta = +\pi, \end{aligned}$$

we have

$$\int_{-\pi}^{+\pi} \Phi(\theta) \left(\frac{1}{2} + \Sigma \cos r\theta \cos r\beta\right) d\theta$$

$$= \left\{\int_{-\pi}^{\lambda_1} f_0(\theta) + \int_{\lambda_1}^{\lambda_2} f_1(\theta) + \ldots. + \int_{\lambda_n}^{\pi} f_n(\theta)\right\} \left(\frac{1}{2} + \Sigma \cos r\theta \cos r\beta\right) d\theta,$$

$$\int_{-\pi}^{+\pi} \Phi(\theta)\, \Sigma \sin r\theta \sin r\beta d\theta$$

$$= \left\{\int_{-\pi}^{\lambda_1} f_0(\theta) + \int_{\lambda_1}^{\lambda_2} f_1(\theta) + \ldots + \int_{\lambda_n}^{\pi} f_n(\theta)\right\} (\Sigma \sin r\theta \sin r\beta)\, d\theta,$$

and the question proposed in Art. 36 is completely solved.

65. Example:

Suppose $\Phi(\theta) = 1$, θ ranging from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$.

Since $$\int_{l_1}^{l_2} \cos r\theta d\theta = \frac{\sin rl_2 - \sin rl_1}{r};$$

$$\therefore \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos r\theta d\theta = 0, \text{ if } r \text{ is even,}$$

$$= \frac{2}{r} \sin \frac{r\pi}{2}, \text{ if } r \text{ is odd.}$$

Whence, for all values of β from $-\pi$ to $+\pi$ inclusive,

$$\frac{1}{2}\left\{\int_{\infty\left(-\frac{\pi}{2}-\beta\right)}^{\infty\left(\frac{\pi}{2}-\beta\right)} + \int_{\infty\left(-\frac{\pi}{2}+\beta\right)}^{\infty\left(\frac{\pi}{2}+\beta\right)}\right\} \frac{\sin\phi}{\phi}\, d\phi$$

$$= \frac{\pi}{2} + 2\left(\cos\beta - \frac{1}{3}\cos 3\beta + \frac{1}{5}\cos 5\beta - \ldots\ldots\right).$$

Shewing that, exclusive of values for which $\infty\,(\pm\frac{\pi}{2}\pm\beta)$ is finite,

$$\begin{aligned}
&\text{from } \beta=-\pi \text{ to } -\frac{\pi}{2}, && \frac{1}{2}(0+0)=\frac{\pi}{2}+2\left(\cos\beta-\frac{1}{3}\cos 3\beta+\dots\right),\\
&\dots\ \beta=-\frac{\pi}{2} \text{ to } +\frac{\pi}{2}, && \frac{1}{2}(\pi+\pi)=\frac{\pi}{2}+2\left(\cos\beta-\frac{1}{3}\cos 3\beta+\dots\right),\\
&\dots\ \beta=+\frac{\pi}{2} \text{ to } +\pi; && 0=\frac{\pi}{2}+2\left(\cos\beta-\frac{1}{3}\cos 3\beta+\dots\right).
\end{aligned}$$

There is false discontinuity when $\pm\frac{\pi}{2}\pm\beta$ vanishes.

If $\beta=\frac{\pi}{2}+\frac{c}{\infty}$, we have

$$\frac{1}{2}\left(\int_{-\infty}^{-c}+\int_{c}^{+\infty}\right)\frac{\sin\phi}{\phi}\,d\phi=\int_{c}^{\infty}\frac{\sin\phi}{\phi}\,d\phi$$
$$=\frac{\pi}{2}+2\left\{\cos\left(\frac{\pi}{2}+\frac{c}{\infty}\right)-\frac{1}{3}\cos 3\left(\frac{\pi}{2}+\frac{c}{\infty}\right)+\dots\right\},$$

the number of terms of the series being $\left(\infty-\frac{1}{2}\right)$.

66. Example:

Suppose $\Phi(\theta)=\theta$, from $\theta=-\pi$ to $\theta=+\pi$.

$$\int_{-\pi}^{+\pi}\theta\sin r\theta\,d\theta=\Bigg|_{-\pi}^{+\pi}\frac{-r\theta\cos r\theta+\sin r\theta}{r^2}=-\frac{2\pi\cos r\pi}{r}.$$

So that

$$\frac{1}{2}\left\{(2\pi+\beta)\int_{-\infty(3\pi+\beta)}^{-\infty(\pi+\beta)}-(2\pi-\beta)\int_{-\infty(3\pi-\beta)}^{-\infty(\pi-\beta)}+\beta\int_{-\infty(\pi+\beta)}^{\infty(\pi-\beta)}\right.$$
$$\left.+\beta\int_{\infty(-\pi+\beta)}^{\infty(\pi+\beta)}+(-2\pi+\beta)\int_{\infty(\pi-\beta)}^{\infty(3\pi-\beta)}+(-2\pi-\beta)\int_{\infty(\pi+\beta)}^{\infty(3\pi+\beta)}\right\}\frac{\sin\phi}{\phi}\,d\phi$$
$$=2\pi\left(\sin\beta-\frac{1}{2}\sin 2\beta+\frac{1}{3}\sin 3\beta-\dots\dots\right).$$

There is false discontinuity when $\beta = -\pi$, and $\beta = +\pi$. Exclusive of values for which $x(\pi \pm \beta)$ is finite, we have, β ranging from $-\pi$ to $+\pi$,

$$\pi\beta = 2\pi\left(\sin\beta - \frac{1}{2}\sin 2\beta + \frac{1}{3}\sin 3\beta - \ldots\ldots\right).$$

If $\beta = \pi - \dfrac{c}{\infty}$, we have

$$2\pi\left\{\sin\left(\pi - \frac{c}{\infty}\right) - \frac{1}{2}\sin 2\left(\pi - \frac{c}{\infty}\right) + \ldots \text{ to } \left(\infty - \frac{1}{2}\right) \text{ terms}\right\}$$

$$= \frac{1}{2}\left\{-\left(\pi + \frac{c}{\infty}\right)\int_{-\infty}^{-c} + \left(\pi - \frac{c}{\infty}\right)\int_{-\infty}^{c} + \left(\pi - \frac{c}{\infty}\right)\int_{-c}^{\infty}\right.$$

$$\left. - \left(\pi + \frac{c}{\infty}\right)\int_{c}^{\infty}\right\}\frac{\sin\phi}{\phi}\,d\phi = 2\pi\int_0^c \frac{\sin\phi}{\phi}\,d\phi,$$

which is π^2 when c is infinite, 0 when c is 0, and has for its greatest maximum value $2\pi\int_0^\pi \dfrac{\sin\phi}{\phi}\,d\phi$.

67. Example:

Suppose $\quad \Phi(\theta) = \theta,\;$ from $\theta = 0$ to $\theta = \alpha$,

and $\quad = -\dfrac{\alpha\theta}{\pi - \alpha} + \dfrac{\pi\alpha}{\pi - \alpha},\;$ from $\theta = \alpha$ to $\theta = \pi$.

Then

$$\int_0^\alpha \theta\sin r\theta\,d\theta = \frac{\sin r\alpha - r\alpha\cos r\alpha}{r^2};\quad \int_\alpha^\pi\left(-\frac{\alpha\theta}{\pi - \alpha} + \frac{\pi\alpha}{\pi - \alpha}\right)\sin r\theta\,d\theta$$

$$= -\frac{\alpha}{\pi - \alpha}\left(\frac{-r\pi\cos r\pi}{r^2} - \frac{\sin r\alpha - r\alpha\cos r\alpha}{r^2}\right) + \frac{\pi\alpha}{\pi - \alpha}\left(-\frac{\cos r\pi}{r} + \frac{\cos r\alpha}{r}\right)$$

$$= \frac{\alpha}{\pi - \alpha}\left(\frac{\sin r\alpha - r\alpha\cos r\alpha + \pi r\cos r\alpha}{r^2}\right);$$

$$\int_0^\pi \Phi(\theta)\sin r\theta\,d\theta = \left(\int_0^\alpha + \int_\alpha^\pi\right) = \frac{\pi\sin r\alpha}{(\pi - \alpha)\,r^2}.$$

So that

$$\frac{\pi}{\pi-\alpha}\left(\sin\alpha\sin\beta+\frac{1}{2^2}\sin 2\alpha\sin 2\beta+\frac{1}{3^2}\sin 3\alpha\sin 3\beta+\ldots\ldots\right)$$

$$=\frac{1}{2}\left\{\beta\int_{-a\beta}^{a(\alpha-\beta)}+\beta\int_{a\beta}^{a(\alpha+\beta)}+\frac{\alpha(\pi-\beta)}{\pi-\alpha}\int_{a(\alpha-\beta)}^{a(\pi-\beta)}\right.$$

$$\left.-\frac{\alpha(\pi+\beta)}{\pi-\alpha}\int_{a(\alpha+\beta)}^{a(\pi+\beta)}\right\}\frac{\sin\phi}{\phi}\,d\phi$$

$$=\frac{\pi\beta}{2},\ \text{from } \beta=0 \text{ to } \beta=\alpha,$$

$$=\frac{\pi}{2}\cdot\frac{\alpha(\pi-\beta)}{\pi-\alpha},\ \text{from } \beta=\alpha \text{ to } \beta=\pi.$$

A change in the sign of β changes the sign of the series.

If $\alpha=\pi-\frac{c}{a}$, the expansion is the same as that in 66 as long as $\frac{rc}{a}$ vanishes. For $\dfrac{\sin r\left(\pi-\frac{c}{a}\right)}{\frac{r^2c}{a}}=-\dfrac{\cos r\pi}{r}$. The expansions are then equal in value, except in the neighbourhood of the false discontinuity.

68. Example:

Suppose $\Phi(\theta)=b\left(1-\frac{\theta}{a}\right)$ from $\theta=0$ to $\theta=\alpha$,

$=0$ .... α ...... π,

and not to change sign with θ.

$$\int_0^\pi\Phi(\theta)\cos r\theta d\theta=\left(\int_0^\alpha+\int_0^\pi\right)\Phi(\theta)\cos r\theta d\theta=b\int_0^\alpha\cos r\theta d\theta\left(1-\frac{\theta}{a}\right)$$

$$=\frac{b\sin r\alpha}{r}-\frac{b}{a}\cdot\frac{r\alpha\sin r\alpha+\cos r\alpha}{r^2}+\frac{b}{r^2a}=\frac{b(1-\cos r\alpha)}{r^2a}.$$

So that

$$\int_0^\pi \Phi(\theta)\left(\tfrac{1}{2}+\Sigma\cos r\theta\cos r\beta\right)d\theta$$

$$=\frac{b\alpha}{4}+b\left(\frac{1-\cos\alpha}{\alpha}\cos\beta+\frac{1-\cos 2\alpha}{2^2\alpha}\cos 2\beta+\dots\right)$$

$$=\frac{1}{2}\left\{b\left(1-\frac{\beta}{\alpha}\right)\int_{-a\beta}^{a(\alpha-\beta)}+b\left(1+\frac{\beta}{\alpha}\right)\int_{a\beta}^{a(\alpha+\beta)}\right\}\frac{\sin\phi}{\phi}d\phi$$

$$=\frac{\pi b}{2}\left(1-\frac{\beta}{\alpha}\right),\qquad\text{from }\beta=0\text{ to }\beta=\alpha,$$

$$=0,\qquad\dots\quad\beta=\alpha\text{ to }\beta=\pi.$$

This result is still true when α is diminished indefinitely, provided $a\alpha$ still remain infinite. If $a\alpha$ were finite, the condition of Art. 60 would be violated, for $\int_0^\alpha\left(-\frac{b}{\alpha}\int_0^\theta\frac{\sin a\theta}{\sin\frac{1}{2}\theta}d\theta\right)d\theta$ will not vanish.

69. Example. In (68) supposing $a\alpha$ finite, we can proceed as follows.

Since

$$\int_0^\theta\left(\frac{1}{2}+\Sigma\cos r\theta\cos r\beta\right)d\theta$$

$$=\frac{1}{2}\int_0^\theta\left\{\frac{1}{2}+\Sigma\cos r(\theta-\beta)+\frac{1}{2}+\Sigma\cos r(\theta+\beta)\right\}d\theta$$

$$=\frac{1}{4}\int_0^\theta\left\{\frac{\sin a(\theta-\beta)}{\sin\frac{1}{2}(\theta-\beta)}+\frac{\sin a(\theta+\beta)}{\sin\frac{1}{2}(\theta+\beta)}\right\}d\theta=\frac{1}{4}\left(\int_{-\beta}^{\theta-\beta}+\int_{+\beta}^{\theta+\beta}\right)\frac{\sin a\theta}{\sin\frac{1}{2}\theta}d\theta$$

$$=\frac{1}{2}\left(\int_{-a\beta}^{a(\theta-\beta)}+\int_{a\beta}^{a(\theta+\beta)}\right)\frac{\sin\phi}{\phi}d\phi,$$

and

$$\frac{d}{d\theta}\left(\int_{-a\beta}^{a(\theta-\beta)}+\int_{a\beta}^{a(\theta+\beta)}\right)\int_0^\theta\frac{\sin\phi}{\phi}d\phi\,d\theta$$

$$=a\int_0^{a(\theta-\beta)}\frac{\sin\phi}{\phi}d\phi+a\int_0^{a(\theta+\beta)}\frac{\sin\phi}{\phi}d\phi$$

$$=a\left(\int_{-a\beta}^{a(\theta-\beta)}+\int_{a\beta}^{a(\theta+\beta)}\right)\frac{\sin\phi}{\phi}d\phi,$$

for $\int_0^{a\beta} \frac{\sin\phi}{\phi}\, d\phi = \int_{-a\beta}^{0} \frac{\sin\phi}{\phi}\, d\phi$; we have,

$$\int_0^\theta\int_0^\theta \left(\frac{1}{2} + \Sigma \cos r\theta \cos r\beta\right) d\theta = \frac{1}{2a}\left(\int_{-a\beta}^{a(\theta-\beta)} + \int_{a\beta}^{a(\theta+\beta)}\right)\int_0^\theta \frac{\sin\phi}{\phi}\, d\phi\, d\theta.$$

Now

$$\int_0^\alpha \Phi(\theta) f(\theta)\, d\theta = \Big|_0^\alpha \Phi(\theta) \int_0^\theta f(\theta)\, d\theta - \Big|_0^\alpha \Phi'(\theta) \left(\int_0^\theta\right)^2 f(\theta)\, d\theta^2,$$

if $\Phi''(\theta) = 0$, which it is in the case considered in (68).

So we have

$$\int_0^\alpha \Phi(\theta) \left(\frac{1}{2} + \Sigma \cos r\theta \cos r\beta\right) d\theta$$

$$= \frac{b}{2a\alpha}\left\{\int_{-a\beta}^{a(\alpha-\beta)} + \int_{a\beta}^{a(\alpha+\beta)}\right\}\int_0^\theta \frac{\sin\phi}{\phi}\, d\phi\, d\theta.$$

Further Examples will be given in the Appendix.

70. PROP. $\frac{1}{2}\int_{x_0}^{x_1} \frac{(e^c - e^{-c})\, dx}{e^c - 2\cos(x+\xi) + e^{-c}}$

$$= \tan^{-1}\left(k \tan\frac{x_1+\xi}{2}\right) - \tan^{-1}\left(k \tan\frac{x_0+\xi}{2}\right),$$

where $k = \frac{e^{\frac{c}{2}} + e^{-\frac{c}{2}}}{e^{\frac{c}{2}} - e^{-\frac{c}{2}}}$. It is supposed that no odd multiple of π occurs between $x_0 + \xi$ and $x_1 + \xi_1$.

In fact,

$$\int \frac{(e^c - e^{-c})\, dx}{e^c - 2\cos(x+\xi) + e^{-c}} = \int \frac{2(e^c - e^{-c})\, d\tan\frac{x+\xi}{2}}{(e^{\frac{c}{2}} - e^{-\frac{c}{2}})^2 + (e^{\frac{c}{2}} + e^{-\frac{c}{2}})^2 \tan^2\frac{x+\xi}{2}}$$

$$= \text{const.} + \frac{2(e^c - e^{-c})}{(e^{\frac{c}{2}} - e^{-\frac{c}{2}})^2} \cdot \frac{e^{\frac{c}{2}} - e^{-\frac{c}{2}}}{e^{\frac{c}{2}} + e^{-\frac{c}{2}}} \tan^{-1}\left(k \tan\frac{x+\xi}{2}\right).$$

71. **Prop.** In like manner

$$\int_{x_0}^{x_1}\frac{\sin\alpha\,dx}{e^x-2\cos\alpha+e^{-x}}=\tan^{-1}\left(\frac{e^{x_0}+1}{e^{x_0}-1}\tan\frac{\alpha}{2}\right)-\tan^{-1}\left(\frac{e^{x_1}+1}{e^{x_1}-1}\tan\frac{\alpha}{2}\right).$$

In fact

$$\frac{d}{dx}\left\{\tan^{-1}\left(\frac{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}\tan\frac{\alpha}{2}\right)\right\}$$

$$=\frac{\tan\frac{\alpha}{2}}{1+\left(\dfrac{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}\tan\dfrac{\alpha}{2}\right)^2}\cdot\frac{\frac{1}{2}(e^{\frac{x}{2}}-e^{-\frac{x}{2}})^2-\frac{1}{2}(e^{\frac{x}{2}}+e^{-\frac{x}{2}})^2}{(e^{\frac{x}{2}}-e^{-\frac{x}{2}})^2}$$

$$=\frac{-2\tan\frac{\alpha}{2}\cos^2\frac{\alpha}{2}}{(e^{\frac{x}{2}}-e^{-\frac{x}{2}})^2\cos^2\frac{\alpha}{2}+(e^{\frac{x}{2}}+e^{-\frac{x}{2}})^2\sin^2\frac{\alpha}{2}}=\frac{-\sin\alpha}{e^x-2\cos\alpha+e^{-x}}.$$

72. To trace the curve

$$y=\Big|^{c=0}\int_x^{x+\delta}\frac{(e^c-e^{-c})\,d\theta}{e^c-2\cos(\theta+\alpha)+e^{-c}},$$

it is supposed that $\delta<2\pi$, so that only one odd multiple of π can occur between $x+\alpha$ and $(x+\delta+\alpha)$. (70) shews that, when no odd multiple of π occurs between $x+\alpha$ and $(x+\delta+\alpha)$, (bearing in mind that y cannot change sign,)

$$y=2\Big|^{k=\infty}\left\{\tan^{-1}\left(k\tan\frac{x+\delta+\alpha}{2}\right)-\tan^{-1}\left(k\tan\frac{x+\alpha}{2}\right)\right\}$$

$$=2\Bigg|^{k=\infty}\tan^{-1}\left\{\frac{k\left(\tan\frac{x+\delta+\alpha}{2}-\tan\frac{x+\alpha}{2}\right)}{1+k^2\tan\frac{x+\alpha}{2}\tan\frac{x+\delta+\alpha}{2}}\right\}$$

$$=2\Bigg|^{k=\infty}\tan^{-1}\left\{\frac{k\sin\frac{\delta}{2}}{\cos\frac{x+\delta+\alpha}{2}\cos\frac{x+\alpha}{2}+k^2\sin\frac{x+\alpha}{2}\sin\frac{x+\delta+\alpha}{2}}\right\}.$$

This will always vanish when both $\frac{x+a}{2}$, $\frac{x+\delta+a}{2}$, differ finitely from multiples of π.

When $\sin\frac{x+\delta+a}{2}$ vanishes, let $\pm k\sin\frac{x+\delta+a}{2}=\mu$, μ being $+$, then $y=2\tan^{-1}\frac{1}{\mu}$.

Similarly, when $\sin\frac{x+a}{2}$ vanishes, $y=2\tan^{-1}\frac{1}{\mu_1}$, if

$$\pm k\sin\frac{x+a}{2}=\mu_1.$$

When an odd multiple of π occurs between x and $x+\delta$,

$$y=\int_x^{(2n+1)\pi}+\int_{(2n+1)\pi}^{x+\delta}=\pi+\pi=2\pi,$$

provided x, $(2n+1)\pi$, $x+\delta$, are separated by finite intervals.

In the curve drawn it is supposed that δ is small, but finite (fig. 5).

$OP_1=2\pi-a-\delta$, $P_1P_2=\pi$, $P_2P_3=\pi$, $OP_6=2\pi-a$, $OP_7=4\pi-a-\delta$, the course of the curve is as represented. When δ vanishes the max. ordinate is still $=2\pi$, provided $k\sin\frac{\delta}{2}$ remain infinite.

If $k\sin\frac{\delta}{2}$ is finite, the max. ordinate becomes $4\tan^{-1}\left(k\tan\frac{\delta}{2}\right)$.

If δ vanishes, $k\sin\frac{\delta}{2}$ remaining infinite, the curve will assume the shape given (fig. 6).

If $\frac{2\pi-\delta}{c}=0$, $y=2\pi$. If $\frac{2\pi-\delta}{c}$ is infinite, although $(2\pi-\delta)$ vanishes, $y=2\pi$; except when x is in the immediate neighbourhood of $(2r\pi-a)$, when y is smaller, having zero for its minimum value. If $\frac{2\pi-\delta}{c}$ is finite, we have, when $x=2r\pi-a$,

$$y=\int_{-a}^{-a+\delta}=\int_{-a}^{\pi-a}+\int_{\pi-a}^{-a+\delta}=4\tan^{-1}\left(k\tan\frac{\delta}{2}\right).$$

73. To trace the curve

$$y = \Big|^{\alpha=0} \int_x^{x+\delta} \frac{\sin \alpha dx}{e^x - 2\cos\alpha + e^{-x}};$$

where δ is $+$, and $\frac{\alpha}{\delta}$ vanishes (vide Art. 71).

From $x = -\infty$ to $x = -\delta - \infty\alpha$, $(\infty\alpha = 0,)$ y is zero;

from $x = -\delta - \infty\alpha$ to $x = -\delta$, $y = \tan^{-1}\frac{\alpha}{x+\delta}$,

for $\int_{-\delta-c\alpha}^{-c\alpha} = \int_{-\delta-c\alpha}^{0} - \int_{-c\alpha}^{0}$; $\Big|^{\alpha=0} \int_{-c\alpha}^{0} \frac{\sin\alpha dx}{e^x - 2\cos\alpha + e^{-x}}$

$$= \frac{\pi}{2} - \Big|^{\alpha=0} \tan^{-1}\left(\frac{e^{\frac{c\alpha}{2}} + e^{-\frac{c\alpha}{2}}}{e^{\frac{c\alpha}{2}} - e^{-\frac{c\alpha}{2}}} \tan\frac{\alpha}{2}\right)$$

$$= \frac{\pi}{2} - \Big|^{\alpha=0} \tan^{-1}\frac{\sec^2\frac{\alpha}{2}}{c} = \frac{\pi}{2} - \tan^{-1}\frac{1}{c};$$

from $x = -\delta$ to $x = -\delta + \infty\alpha$, $y = \pi - \tan^{-1}\frac{\alpha}{x+\delta}$;

from $x = -\delta + \infty\alpha$ to $x = -\infty\alpha$, $y = \pi$;

from $x = -\infty\alpha$ to $x = 0$, $y = \pi - \tan^{-1}\frac{\alpha}{x}$;

from $x = 0$ to $x = \infty\alpha$, $y = \tan^{-1}\frac{\alpha}{x}$;

from $x = \infty\alpha$ to $x = \infty$, $y = 0$.

The course of the curve is $P_1P_2P_3P_4OP_5$ (as in fig. 7).

Suppose δ indefinitely diminished, yet so that $\frac{\delta}{\alpha}$ still remain infinite, the curve will be as before, OP_3 being indefinitely diminished. But when $\frac{\delta}{\alpha}$ becomes finite, the maximum ordinate of the curve becomes

$$2\Big|^{\alpha=0} \int_0^{\frac{\delta}{2}} \frac{\sin\alpha dx}{e^x - 2\cos\alpha + e^{-x}},$$

which, as we have seen,

$$= \pi - 2\tan^{-1}\left(\frac{e^{\frac{\delta}{4}} + e^{-\frac{\delta}{4}}}{e^{\frac{\delta}{4}} - e^{-\frac{\delta}{4}}}\tan\frac{\alpha}{2}\right) = \pi - 2\tan^{-1}\frac{\sec^2\frac{\alpha}{2}}{\left(\frac{\delta}{2\alpha}\right)}$$

$$= \pi - 2\tan^{-1}\frac{2\alpha}{\delta},$$

which vanishes when $\frac{\delta}{\alpha}$ is indefinitely small.

74. Prop. Suppose $F(v)$ is a function of v which is always finite as v increases from x to $(x+2l)$, and is such that

$$\left\{F\left(v+\frac{2l}{n}\right) - F(v)\right\}$$

can be made less than any assignable quantity by taking n large enough, then

$$\Big|^{nc=0}\int_x^{x+2l}\frac{(e^{nc}-e^{-nc})\,F(v)\,dv}{e^{nc} - 2\cos\frac{\pi(v-x)}{l} + e^{-nc}} = l\{F(x) + F(x+2l)\}.$$

The resolution of $\left\{e^{nc} - 2\cos\frac{\pi(v-x)}{l} + e^{-nc}\right\}$ into n factors, when n is an integer, gives us

$$e^{nc} - 2\cos\frac{\pi(v-x)}{l} + e^{-nc}$$

$$= \overset{r=n}{\underset{r=1}{P}}\left[e^c - 2\cos\left\{\frac{2(r-1)\pi}{n} + \frac{\pi(v-x)}{nl}\right\} + e^{-c}\right].$$

Taking the logarithm of both sides, and differentiating with respect to c, we get

$$\frac{n(e^{nc}-e^{-nc})}{e^{nc} - 2\cos\frac{\pi(v-x)}{l} + e^{-nc}} = \sum_{r=1}^{r=n}\frac{e^c - e^{-c}}{e^c - 2\cos\left\{\frac{2(r-1)\pi}{n} + \frac{\pi(v-x)}{nl}\right\} + e^{-c}}.$$

Now, from Art. 70, we obtain the following definite integrals :

$$\int_x^{\frac{2l}{n}+x} \frac{(e^c - e^{-c})\,dv}{n\left\{e^c - 2\cos\dfrac{\pi(v-x)}{nl} + e^{-c}\right\}}$$

$$= \frac{2l}{\pi}\left\{\tan^{-1}\left(k\tan\frac{\pi}{n^2}\right) - \tan^{-1} 0\right\} = \frac{2l}{\pi}\tan^{-1}\left(k\tan\frac{\pi}{n^2}\right),$$

$$\int_{x+2l-\frac{2l}{n}}^{x+2l} \frac{(e^c - e^{-c})\,dv}{n\left\{e^c - 2\cos\dfrac{2(n-1)\,l\pi + \pi(v-x)}{nl} + e^{-c}\right\}}$$

$$= \frac{2l}{\pi}\left[\pi - \tan^{-1}\left\{k\tan\left(\pi - \frac{\pi}{n^2}\right)\right\} = \frac{2l}{\pi}\tan^{-1}\left(k\tan\frac{\pi}{n^2}\right)\right],$$

$$\int_x^{x+2l} \frac{(e^{nc} - e^{-nc})\,dv}{e^{nc} - 2\cos\dfrac{\pi(v-x)}{l} + e^{-nc}} = 2l.$$

Let A be the least, and Ω the greatest, value which $F(v)$ receives, as v increases from x to $(x+2l)$; then, $\theta_1, \theta_2, \ldots$ representing quantities between zero and unity,

$$\int_x^{\frac{2l}{n}+x} \frac{(e^c - e^{-c})\,F(v)\,dv}{n\left\{e^c - 2\cos\dfrac{\pi(v-x)}{nl} + e^{-c}\right\}} = \frac{2l}{\pi}F\left(x + \frac{2\theta_1 l}{n}\right)\tan^{-1}\left(k\tan\frac{\pi}{n^2}\right),$$

$$\int_{x+2l-\frac{2l}{n}}^{x+2l} \frac{(e^c - e^{-c})\,F(v)\,dv}{n\left\{e^c - 2\cos\dfrac{2(n-1)\,l\pi + \pi(v-x)}{nl} + e^{-c}\right\}}$$

$$= \frac{2l}{\pi}F\left(x + 2l - \frac{2\theta_2 l}{n}\right)\tan^{-1}\left(k\tan\frac{\pi}{n^2}\right),$$

$$\int_{\frac{2l}{n}+x}^{x+2l} \frac{(e^c - e^{-c})\,F(v)\,dv}{n\left\{e^c - 2\cos\dfrac{\pi(v-x)}{nl} + e^{-c}\right\}}$$

$$\begin{matrix} > A \\ < \Omega \end{matrix} \int_{\frac{2l}{n}+x}^{x+2l} \frac{(e^c - e^{-c})\,dv}{n\left\{e^c - 2\cos\dfrac{\pi(v-x)}{nl} + e^{-c}\right\}},$$

$$\int_x^{x+2l-\frac{2l}{n}} \frac{(e^c-e^{-c})\,F(v)\,dv}{n\left\{e^c-2\cos\dfrac{2(n-1)\,l\pi+\pi(v-x)}{nl}+e^{-c}\right\}}$$

$$\begin{matrix}>A\\<\Omega\end{matrix}\int_x^{x+2l-\frac{2l}{n}} \frac{(e^c-e^{-c})\,dv}{n\left\{e^c-2\cos\dfrac{2(n-1)\,l\pi+\pi(v-x)}{nl}+e^{-c}\right\}}.$$

$$\int_x^{x+2l}\left[\frac{e^{nc}-e^{-nc}}{e^{nc}-2\cos\dfrac{\pi(v-x)}{l}+e^{-nc}}-\frac{e^c-e^{-c}}{n\left\{e^c-2\cos\dfrac{\pi(v-x)}{nl}+e^{-c}\right\}}\right.$$
$$\left.-\frac{e^c-e^{-c}}{n\left\{e^c-2\cos\dfrac{2(n-1)\,l\pi+\pi(v-x)}{nl}+e^{-c}\right\}}\right]F(v)\,dv.$$

$$\begin{matrix}>A\\<\Omega\end{matrix}\int_x^{x+2l}\left[\frac{e^{nc}-e^{-nc}}{e^{nc}-2\cos\dfrac{\pi(v-x)}{l}+e^{-nc}}-\frac{e^c-e^{-c}}{n\left\{e^c-2\cos\dfrac{\pi(v-x)}{nl}+e^{-c}\right\}}\right.$$
$$\left.-\frac{e^c-e^{-c}}{n\left\{e^c-2\cos\dfrac{2(n-1)\,l\pi+\pi(v-x)}{nl}+e^{-c}\right\}}\right]dv;$$

whence, by addition,

$$\int_x^{x+2l}\frac{(e^{nc}-e^{-nc})\,F(v)\,dv}{e^{nc}-2\cos\dfrac{\pi(v-x)}{l}+e^{-nc}}\begin{matrix}>A\\<\Omega\end{matrix}\left[\int_x^{x+2l}\frac{(e^{nc}-e^{-nc})\,dv}{e^{nc}-2\cos\dfrac{\pi(v-x)}{l}+e^{-nc}}\right.$$
$$-\int_x^{\frac{2l}{n}+x}\frac{(e^c-e^{-c})\,dv}{n\left\{e^c-2\cos\dfrac{\pi(v-x)}{nl}+e^{-c}\right\}}$$
$$\left.-\int_{x+2l-\frac{2l}{n}}^{x+2l}\frac{(e^c-e^{-c})\,dv}{n\left\{e^c-2\cos\dfrac{2(n-1)\,l\pi+\pi(v-x)}{nl}+e^{-c}\right\}}\right]$$
$$+\frac{2l}{\pi}\left\{F\left(x+\frac{2\theta_1 l}{n}\right)+F\left(x+2l-\frac{2\theta_2 l}{n}\right)\right\}\tan^{-1}\left(k\tan\frac{\pi}{n^2}\right)$$

$$\begin{matrix}>A\\<\Omega\end{matrix}\left\{2l-\frac{4l}{\pi}\tan^{-1}\left(k\tan\frac{\pi}{n^2}\right)\right\}+\frac{2l}{\pi}\left\{F\left(x+\frac{2\theta_1 l}{n}\right)\right.$$
$$\left.+F\left(x+2l-\frac{2\theta_2 l}{n}\right)\right\}\tan^{-1}\left(k\tan\frac{\pi}{n^2}\right).$$

Now $$\Bigg|^{c=0} \frac{e^{\frac{c}{2}}+e^{-\frac{c}{2}}}{e^{\frac{c}{2}}-e^{-\frac{c}{2}}} \tan\frac{\pi c^\theta}{a} = 2 \Bigg|^{c=0} \frac{\frac{\pi\theta c^{\theta-1}}{a}\sec^2\frac{\pi c^\theta}{a}}{\frac{1}{2}\left(e^{\frac{c}{2}}+e^{-\frac{c}{2}}\right)} = \Bigg|^{c=0} \frac{2\pi\theta c^{\theta-1}}{a},$$

which is infinite, if θ is a fraction neither infinitely near zero nor infinitely near unity.

Now, although n is truly discontinuous, yet if c be so taken that $c^\theta n^2$ be always finite, n may be taken so great that $k\tan\frac{\pi}{n^2}$ be infinite, and $\tan^{-1}\left(k\tan\frac{\pi}{n^2}\right)$ have $\frac{\pi}{2}$ for its limit.

We have, then, since

$$\left\{F\left(x+\frac{2\theta_1 l}{n}\right)-F(x)\right\},\quad \left\{F\left(x+2l-\frac{2\theta_2 l}{n}\right)-F(x+2l)\right\},$$

both by hypothesis vanish,

$$\Bigg|^{nc=0} \int_x^{x+2l} \frac{(e^{nc}-e^{-nc})\,F(v)\,dv}{e^{nc}-2\cos\frac{\pi(v-x)}{l}+e^{-nc}} = l\,\{F(x)+F(x+2l)\}.$$

75. **Prop.** Subject to the same limitations as in (74) we have

$$\Bigg|^{nc=0} \int_u^{u+2l} \frac{(e^{nc}-e^{-nc})\,F(v)\,dv}{e^{nc}-2\cos\frac{\pi(v-x)}{l}+e^{-nc}} = 2l\,F(x),$$

provided also that $\left\{e^{nc}-2\cos\frac{\pi(u-x)}{l}+e^{-nc}\right\}$ does not vanish; x is supposed intermediate to u and $(u+2l)$. If x be not so, in place of $2lF(x)$ we must write $2lF(x+2rl)$, where $(x+2rl)$ is intermediate to u and $(u+2l)$.

76. Subject to the same limitations as in (74),

$$\Bigg|^{nc=0} \int_{x+\delta}^{x+\delta+2l} \frac{(e^{nc}-e^{-nc})\,F(v)\,dv}{e^{nc}-2\cos\frac{\pi(v-x)}{l}+e^{-nc}} = lF(x)\left\{1-\frac{2}{\pi}\tan^{-1}\left(k\tan\frac{\delta}{2}\right)\right\}$$
$$+\,lF(2l+x)\left\{1+\frac{2}{\pi}\tan^{-1}\left(k\tan\frac{\delta}{2}\right)\right\}.$$

For (Art. 70)

$$\int_x^{x+\delta} \frac{(e^{nc} - e^{-nc}) F(v)\, dv}{e^{nc} - 2\cos\frac{\pi(v-x)}{l} + e^{-nc}} = \frac{2l}{\pi} F(x) \tan^{-1}\left(k \tan\frac{\delta}{2}\right),$$

and $$\int_{x+2l}^{x+\delta+2l} \frac{(e^{nc} - e^{-nc}) F(v)\, dv}{e^{nc} - 2\cos\frac{\pi(v-x)}{l} + e^{-nc}} = \frac{2l}{\pi} F(x+2l) \tan^{-1}\left(k \tan\frac{\delta}{2}\right).$$

Whence the proposition.

We observe that the definite integral has false discontinuity, except when $F(x) = F(2l + x)$, but is never truly discontinuous.

77. Poisson, Tait and Thomson, and others, have attempted, by means of the definite integral discussed in Arts. 74 to 76, to establish formulæ corresponding to our formulæ of Art. 67. Generally speaking, an indirect method of establishing a proposition is either redundant, or else contains a concealed *petitio principii.* In the case before us, we have, by multiplication,

$$\left\{e^{nc} - 2\cos\frac{\pi(v-x)}{l} + e^{-nc}\right\}\left\{1 + 2e^{-nc}\cos\frac{\pi(v-x)}{l}\right.$$

$$\left. + 2e^{-2nc}\cos\frac{2\pi(v-x)}{l} + \ldots + 2e^{-snc}\cos\frac{s\pi(v-x)}{l}\right\}$$

$$= e^{nc} + 2\cos\frac{\pi(v-x)}{l} + 2e^{-nc}\cos\frac{2\pi(v-x)}{l} + \ldots$$

$$+ 2e^{-(s-1)nc}\cos\frac{s\pi(v-x)}{l} - 2\cos\frac{\pi(v-x)}{l} - 4e^{-nc}\cos^2\frac{\pi(v-x)}{l} - \ldots$$

$$- 4e^{-(s-1)nc}\cos\frac{(s-1)\pi(v-x)}{l}\cos\frac{\pi(v-x)}{l} + e^{-nc} + \ldots$$

$$+ 2e^{-(s-1)nc}\cos\frac{(s-2)\pi(v-x)}{l} - 4e^{-snc}\cos\frac{s\pi(v-x)}{l}\cos\frac{\pi(v-x)}{l}$$

$$+ 2e^{-snc}\cos\frac{(s-1)\pi(v-x)}{l} + 2e^{-(s+1)nc}\cos\frac{s\pi(v-x)}{l}$$

$$= e^{nc} - e^{-nc} - 2e^{-snc}\cos\frac{(s+1)\pi(v-x)}{l} + 2e^{-(s+1)nc}\cos\frac{s\pi(v-x)}{l}.$$

So that

$$\int_u^{u+2l} \frac{(e^{nc}-e^{-nc})\,F(v)\,dv}{e^{nc}-2\cos\frac{\pi(v-x)}{l}+e^{-nc}}$$

$$=\int_u^{u+2l} F(v)\,dv+2\sum_1^s\int_u^{u+2l} e^{-mnc}\cos\frac{s\pi(v-x)}{l}\,F(v)\,dv$$

$$+\,2e^{-(s+\frac{1}{2})nc}\int_u^{u+2l}\frac{e^{\frac{1}{2}nc}\cos\frac{(s+1)\pi(v-x)}{l}-e^{-\frac{1}{2}nc}\cos\frac{s\pi(v-x)}{l}}{e^{nc}-2\cos\frac{\pi(v-x)}{l}+e^{-nc}}\,F(v)\,dv.$$

Now $$\int_u^{u+2l}\frac{e^{\frac{1}{2}nc}\cos\frac{(s+1)\pi(v-x)}{l}-e^{-\frac{1}{2}nc}\cos\frac{s\pi(v-x)}{l}}{e^{nc}-2\cos\frac{\pi(v-x)}{l}+e^{-nc}}\,F(v)\,dv$$

$$=\int_u^{u+2l}\frac{(e^{\frac{1}{2}nc}-e^{-\frac{1}{2}nc})\cos\frac{(s+\frac{1}{2})\pi(v-x)}{l}\cos\frac{\frac{1}{2}\pi(v-x)}{l}}{e^{nc}-2\cos\frac{\pi(v-x)}{l}+e^{-nc}}\,F(v)\,dv$$

$$-\int_u^{u+2l}\frac{(e^{\frac{1}{2}nc}+e^{-\frac{1}{2}nc})\sin\frac{(s+\frac{1}{2})\pi(v-x)}{l}\sin\frac{\frac{1}{2}\pi(v-x)}{l}}{e^{nc}-2\cos\frac{\pi(v-x)}{l}+e^{-nc}}\,F(v)\,dv,$$

the first of which, by 75, generally

$$=lF(x)\cos\frac{(s+\frac{1}{2})\pi(x-x)}{l}\cos\frac{\frac{1}{2}\pi(x-x)}{l},$$

and the evaluation of the other involves the discussion of the integral

$$\int_u^{u+2l}\frac{\sin\frac{(s+\frac{1}{2})\pi(v-x)}{l}\,F(v)\,dv}{\sin\frac{\pi(v-x)}{2l}},$$

which is the envelop of the series.

78. In many cases when the methods of Arts. 74 to 76 fail, owing to the form of $F(v)$, the formulæ of Arts. 54, 55 will serve to give the value of the integral. An example of this is furnished in Art. 69.

79. PROP.
$$\Big|^{\alpha=0} \int_{x_1}^{x_2} \frac{\sin\alpha \,.\, F(x)\,dx}{e^{x}-2\cos\alpha+e^{-x}}$$
$$= \Big|^{\alpha=0} F(0) \int_{x_1}^{x_2} \frac{\sin\alpha dx}{e^{x}-2\cos\alpha+e^{-x}},$$

provided $\int_{x_1}^{x} \sqrt{F'(x)^2}\,dx$ is never infinite from $x=x_1$ to $x=x_2$, and that, for all finite values of c, $\int_0^{c\alpha} \sqrt{F'(x)^2}\,dx$ may be made to differ from $F(0)$ by less than any assignable quantity by taking α small enough.

We have (Art. 54)

$$\int_{x_1}^{x} \frac{\sin\alpha \,.\, F(x)\,dx}{e^{x}-2\cos\alpha+e^{-x}} \equiv F(x) \int_{x_1}^{x} \frac{\sin\alpha dx}{e^{x}-2\cos\alpha+e^{-x}}$$
$$-\int_{x_1}^{x} \left(F'(x) \int_{x_1}^{x} \frac{\sin\alpha dx}{e^{x}-2\cos\alpha+e^{-x}}\right) dx.$$

Now, if x_1, x have the same sign, and $\frac{\alpha}{x_1}$, $\frac{\alpha}{x}$ vanish,

$$\int_{x_1}^{x} \frac{\sin\alpha dx}{e^{x}-2\cos\alpha+e^{-x}},$$

which is always +, can be made less than any assignable quantity by taking α small enough, and the square root of the square of

$$\int_{x_1}^{x} \left(F'(x) \int_{x_1}^{x} \frac{\sin\alpha dx}{e^{x}-2\cos\alpha+e^{-x}}\right) dx$$
$$< \int_{x_1}^{x} \frac{\sin\alpha dx}{e^{x}-2\cos\alpha+e^{-x}} \int_{x_1}^{x} \sqrt{F'(x)^2}\,dx,$$

and $\int_{x_1}^{x} \sqrt{F'(x)^2}\,dx$ is finite by hypothesis: so the last term in the above identity vanishes.

Again, if $\frac{\alpha}{x_1}$, $\frac{\alpha}{x}$ are finite, the square root of the square of

$$\int_{x_1}^{x}\left(F'(x)\int_{x_1}^{x}\frac{\sin\alpha dx}{e^{x}-2\cos\alpha+e^{-x}}\right)dx<\pi\int_{x_1}^{x}\sqrt{F'(x)^2}\,dx,$$

an evanescent quantity: and, by hypothesis, in this case $F(x)=F(0)$.

All other cases are readily demonstrated by help of the above two and the axiomatic principle that if a quantitative proposition is true for all finite values of a variable, it is also true for some infinite. In other words, if a proposition is not true for any infinite values of a variable, there are also some finite values of the variable for which it is not true.

The principle, as above stated, is axiomatic in its nature, and will be conceded by every one, as far as it is necessary for this proposition. The word 'quantitative' is above introduced to meet paradoxical reasoning such as 'all finite values of a variable are finite, therefore some infinite values of a variable are finite.'

80. In cases where the conditions, stated in last Art. as necessary, do not hold, we can often find a value of x so that,

$$\left(\phi(x)\equiv\Big|^{\alpha=0}\frac{\sin\alpha}{e^{x}-2\cos\alpha+e^{-x}}\right),$$

$$\int_0^x F^{(n+1)}(x)\left\{\left(\int_0^x\right)^{n+1}\phi(x)\,dx^{n+1}\right\}dx=0,$$

and in that case (Art. 54),

$$\int_{x_1}^{x_2}F(x)\,\phi(x)\,dx=F(0)\int_{x_1}^{x_2}\phi(x)\,dx$$

$$-F'(0)\int_{x_1}^{x_2}\left\{\int_0^x\phi(x)\,dx\right\}dx+\ldots\ldots$$

$$+(-1)^n F^{(n)}(0)\int_{x_1}^{x_2}\left\{\left(\int_0^x\right)^n\phi(x)\,dx^n\right\}dx.$$

This formula can always be applied when

$$F(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + F_n(x),$$

and $F_n(x)$ satisfies the condition stated as necessary in Art. 79. It is not at all necessary that a_0, $a_1, \ldots a_n$, be finite.

81. Since, X_1, X_2 being functions of x,

$$\int_{X_1}^{X_2} \frac{\sin ay}{\sin \frac{1}{2} y}\, dy = X_2 - X_1 + 2 \Big|_{X_1}^{X_2} \sum_1^s \frac{\sin ry}{r}.$$

We have

$$\int_{x_1}^{x_2} \int_{X_1}^{X_2} \frac{\sin ax \sin ay}{\sin \frac{1}{2} x \sin \frac{1}{2} y}\, dx\, dy$$

$$= \int_{x_1}^{x_2} \frac{(X_2 - X_1) \sin ax\, dx}{\sin \frac{1}{2} x} + 2 \sum_1^s \int_{x_1}^{x_2} \frac{(\sin rX_2 - \sin rX_1) \sin ax}{r \sin \frac{1}{2} x}\, dx.$$

Now, if

$$\int_{x_1}^{x_2} \sqrt{\frac{dX_2^2}{dx}}\, dx,\quad \int_{x_1}^{x_2} \sqrt{\cos^2 r X_2 \frac{dX_2^2}{dx}}\, dx,$$

and corresponding quantities for X_1, are always finite, and

$$\int_0^{\pm\delta} \sqrt{\frac{dX_2^2}{dx}}\, dx,\quad \int_0^{\pm\delta} \sqrt{\frac{dX_1^2}{dx}}\, dx,$$

evanescent, (i.e. if both X_1, X_2, satisfy the tests of Art. 60), we can apply the formulæ of that Article, and so obtain

$$\int_{x_1}^{x_2} \int_{X_1}^{X_2} \frac{\sin ax \sin ay}{\sin \frac{1}{2} x \sin \frac{1}{2} y}\, dx\, dy = \Big|^{x=0} (X_2 - X_1) \int_{x_1}^{x_2} \frac{\sin ax\, dx}{\sin \frac{1}{2} x}$$

$$+ 2 \sum_1^s \Big|^{x=0} \frac{(\sin rX_2 - \sin rX_1)}{r} \int_{x_1}^{x_2} \frac{\sin ax}{\sin \frac{1}{2} x}\, dx$$

$$= \left(\Big|^{x=0} \int_{X_1}^{X_2} \frac{\sin ay}{\sin \frac{1}{2} y}\, dy \right) \int_{x_1}^{x_2} \frac{\sin ax}{\sin \frac{1}{2} x}\, dx.$$

82. **Prop.** Suppose $f(x, y)$ to satisfy the tests of Art. 60, whether considered as a function of x, or of y, and X_1, X_2 also to satisfy those tests: then, when -2π, X_1, X_2, $+2\pi$, are in order for all values of x included within the limits, as also -2π, x_1, x_2, $+2\pi$, and $a(2\pi + X_2)$, $a(2\pi - X_2)$, $a(2\pi + x_1)$, $a(2\pi - x_2)$, are all infinite, shall

$$\Big|^{a=\infty} \int_{x_1}^{x_2} \int_{X_1}^{X_2} \frac{f(x, y) \sin ax \sin ay}{\sin \frac{1}{2} x \sin \frac{1}{2} y} \, dx \, dy$$

$$= 4 f(0, 0) \int_{ax_1}^{ax_2} \frac{\sin \phi}{\phi} \, d\phi \, \Big|^{x=0} \int_{aX_1}^{aX_2} \frac{\sin \phi}{\phi} \, d\phi.$$

We have

$$\int_{X_1}^{X_2} \frac{f(x, y) \sin ax \sin ay}{\sin \frac{1}{2} x \sin \frac{1}{2} y} \, dy = f(x, 0) \frac{\sin ax}{\sin \frac{1}{2} x} \int_{X_1}^{X_2} \frac{\sin ay}{\sin \frac{1}{2} y} \, dy,$$

and

$$\int_{x_1}^{x_2} \{f(x, 0) - f(0, 0)\} \left(\frac{\sin ax}{\sin \frac{1}{2} x} \int_{X_1}^{X_2} \frac{\sin ay}{\sin \frac{1}{2} y} \, dy \right) dx$$

$$= \{f(x_2, 0) - f(0, 0)\} \int_{x_1}^{x_2} \left(\frac{\sin ax}{\sin \frac{1}{2} x} \int_{X_1}^{X_2} \frac{\sin ay}{\sin \frac{1}{2} y} \, dy \right) dx$$

$$- \int_{x_1}^{x_2} \left\{ f'(x, 0) \int_{x_1}^{x_2} \frac{\sin ax}{\sin \frac{1}{2} x} \left(\int_{X_1}^{X_2} \frac{\sin ay}{\sin \frac{1}{2} y} \, dy \right) dx \right\} dx$$

$$= \{f(x_2, 0) - f(0, 0)\} \int_{x_1}^{x_2} \int_{X_1}^{X_2} \frac{\sin ax \sin ay \, dx \, dy}{\sin \frac{1}{2} x \sin \frac{1}{2} y}$$

$$- \int_{x_1}^{x_2} \left\{ f'(x, 0) \int_{x_1}^{x_2} \int_{X_1}^{X_2} \frac{\sin ax \sin ay \, dx \, dy}{\sin \frac{1}{2} x \sin \frac{1}{2} y} \right\} dx.$$

Consider a little square, with side δ, such that δ ultimately vanishes, although $a\delta$ is infinite, and the origin is in the centre of the square. Then, if the limits which bound the surface $\int_{x_1}^{x_2}\int_{X_1}^{X_2} dxdy$ enclose no portion of the little square, it may be shewn, as in Art. 60 firstly, that the right-hand member of the last equation vanishes. And it can be shewn, as in Art. 60 secondly, that the same expression vanishes for the little square, or any portion of it, since X_2, X_1, satisfy the tests. Now (Art. 47),

$$\int_{X_1}^{X_2}\frac{\sin ay}{\sin\frac{1}{2}y}\,dy = 2\int_{aX_1}^{aX_2}\frac{\sin\phi}{\phi}\,d\phi; \qquad \int_{x_1}^{x_2}\frac{\sin ax}{\sin\frac{1}{2}x}\,dx = 2\int_{ax_1}^{ax_2}\frac{\sin\phi}{\phi}\,d\phi;$$

so that

$$\Big|^{a=\infty}\int_{x_1}^{x_2}\int_{X_1}^{X_2}\frac{f(x,\,y)\sin ax\sin ay}{\sin\frac{1}{2}x\sin\frac{1}{2}y}\,dxdy$$

$$= 4f(0,\,0)\int_{ax_1}^{ax_2}\frac{\sin\phi}{\phi}\,d\phi\,\Big|^{x=0}\int_{aX_1}^{aX_2}\frac{\sin\phi}{\phi}\,d\phi.$$

A similar proposition may be established for any number of independent variables, which is here established for two.

83. Formulæ corresponding to those of Art. 64 can be established for any number of independent variables. Thus, to take the case of two independent variables, if we seek the value of

$$\Big|^{a=\infty}\int_{t_1}^{t_2}\int_{l_1}^{l_2}\frac{f(\theta,\,\theta')\sin a(\theta-\beta)\sin a(\theta'-\beta')\,d\theta'd\theta}{\sin\frac{1}{2}(\theta-\beta)\sin\frac{1}{2}(\theta'-\beta')},$$

we find, by Art. 64, that it

$$=\Big|^{a=\infty} 2\int_{t_1}^{t_2}\Bigg\{f(2\pi+\beta,\,\theta')\int_{a(-2\pi+l_1-\beta)}^{a(-2\pi+l_2-\beta)}+f(\beta,\,\theta')\int_{a(l_1-\beta)}^{a(l_2-\beta)}$$

$$+f(-2\pi+\beta,\,\theta')\int_{a(2\pi+l_1-\beta)}^{a(2\pi+l_2-\beta)}\Bigg\}\frac{\sin a\,(\theta'-\beta')}{\sin\frac{1}{2}(\theta'-\beta')}\,\frac{\sin\phi}{\phi}\,d\phi d\theta,$$

$$= \Big|^{a=\infty} 4\Big\{f(2\pi+\beta,\ 2\pi+\beta')\int_{a(-2\pi+l_1-\beta)}^{a(-2\pi+l_2-\beta)} \frac{\sin\phi}{\phi}\,d\phi \,.\int_{a'(-2\pi+l'_1-\beta')}^{a'(-2\pi+l'_2-\beta')} \frac{\sin\phi}{\phi}\,d\phi + f(2\pi+\beta,\ \beta')\iint$$

$$+f(2\pi+\beta,\ -2\pi+\beta')\iint + f(\beta,\ 2\pi+\beta')\iint + f(\beta,\ \beta')\int_{a(l_1-\beta)}^{a(l_2-\beta)}\int_{a(l_1'-\beta')}^{a(l'_2-\beta')}$$

$$+f(\beta,\ -2\pi+\beta')\iint + f(-2\pi+\beta,\ 2\pi+\beta')\iint + f(-2\pi+\beta,\ \beta')\iint$$

$$+f(-2\beta+\beta,\ -2\pi+\beta')\iint\Big\}.$$

The limits of the integrals all follow the same law, and so can readily be determined. The term involving $f(\beta, \beta')$ is the only one of general importance.

84. The last Article enables us to express by means of a double series the equation to a periodic surface every square of which is identical in form with every other square, and has any form which we may please to assign. It should be observed that of the nine terms of which the second member of the last equation consists, one only is generally effective, four others are effective only at the edges, and four others only at the corners, of each square. If the form of f is itself properly periodic, there will be no false discontinuity; if otherwise, the false discontinuity at the edges resembles a plane folded backwards and forwards upon itself an infinite number of times, yet so as not to acquire thickness by the process. But we may select other forms of definite integrals, and so obtain false discontinuity of a simpler kind. Such definite integrals are

$$\Big|^{c=0} \int_{u'}^{u'+2l'}\int_{u}^{u+2l} \frac{(e^c-e^{-c})^2 F(v',\ v)\,dv'dv}{\left(e^c-2\cos\frac{\pi(v'-x')}{l'}+e^{-c}\right)\left(e^c-2\cos\frac{\pi(v-x)}{l}+e^{-c}\right)};$$

$$\Big|^{a=0}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\frac{\sin^2\alpha\,.\,F(x,\ y)\,dxdy}{(e^x-2\cos\alpha+e^{-x})(e^y-2\cos\alpha+e^{-y})};$$

$$\Big|^{a=1}\int_0^{\pi}\int_0^{2\pi}\frac{(1-a)^2F(\theta',\ \psi')\sin\theta' d\theta' d\psi'}{(1-2ap+a^2)^{\frac{3}{2}}},$$

where $\qquad p=\cos\theta\,\cos\theta'+\sin\theta\,\sin\theta'\,\cos(\psi'-\psi).$

(Vide Poisson, *Théorie de la Chaleur.*)

Or we may get the false discontinuity to be simple by suitably choosing for $f(\theta, \theta')$, since $f(\theta, \theta')$ may be falsely discontinuous. The following example is an instance of this.

85. Example:

To find the equation to the surface of a pyramid, periodically repeated, the equation to the four slopes which meet at the vertex on the axis of z being

$$z = \pi - x, \quad z = \pi + x, \quad z = \pi - y, \quad z = \pi + y.$$

The equation will be

$$z = \Big|^{a=\infty} \frac{1}{4\pi^2} \left\{ \left(\int_{-\pi}^{0} \int_{-\pi}^{x} + \int_{0}^{\pi} \int_{-\pi}^{-x} \right) (\pi + y) + \int_{-\pi}^{0} \int_{x}^{-x} (\pi + x) \right.$$

$$\left. + \left(\int_{-\pi}^{0} \int_{-x}^{\pi} + \int_{0}^{\pi} \int_{x}^{\pi} \right) (\pi - y) + \int_{0}^{\pi} \int_{-x}^{x} (\pi - x) \right\}$$

$$\frac{\sin a (x - x') \sin a (y - y')}{\sin \frac{1}{2} (x - x') \sin \frac{1}{2} (y - y')} dxdy.$$

Now $\dfrac{\sin a (x - x') \sin a (y - y')}{\sin \frac{1}{2} (x - x') \sin \frac{1}{2} (y - y')}$

$$= \{1 + 2\sum_{1}^{s} \cos r (x - x')\} \{1 + 2\sum_{1}^{s} \cos r (y - y')\};$$

and $$\int_{x}^{-x} \sin ry dy = 0; \quad \left(\int_{x}^{\pi} + \int_{-\pi}^{-x} \right) \sin ry dy = 0;$$

$$\left(\int_{-\pi}^{-x} - \int_{x}^{\pi} \right) y \sin ry dy = 0;$$

or the series can contain no term involving $\sin ry'$; in like manner it can contain no term involving $\sin rx'$. This is secured by choosing the function such that a change in sign in x' or y' affects not its value.

We have then only to operate upon

$$1+2\sum_1^s \cos ry \cos ry' + 2\sum_1^s \cos r'x \cos rx' + 4\sum_1^s\sum_1^s \cos r'x \cos ry \cos r'x' \cos ry',$$

and our equation will be

$$\frac{1}{4}z = \Big|^{s=\infty} \frac{1}{4\pi^2}\left\{\int_0^\pi\int_x^\pi(\pi-y) + \int_0^\pi\int_0^x(\pi-x)\right\}$$

$$(1+2\sum_1^s \cos r'x \cos r'x')(1+2\sum_1^s \cos ry \cos ry')\, dx\, dy.$$

Now $\quad \frac{1}{\pi^2}\left\{\int_0^\pi\int_x^\pi(\pi-y) + \int_0^\pi\int_0^x(\pi-x)\right\} dx\, dy$

$$= \frac{1}{\pi^2}\int_0^\pi\left(\pi^2 - \pi x - \frac{\pi^2}{2} + \frac{x^2}{2} + \pi x - x^2\right) dx$$

$$= \frac{1}{\pi^2}\int_0^\pi\left(\frac{\pi^2}{2} - \frac{x^2}{2}\right) dx = \pi\left(\frac{1}{2} - \frac{1}{6}\right) = \frac{\pi}{3}.$$

$$\frac{2\cos ry'}{\pi^2}\left\{\int_x^\pi (\pi - y)\cos ry\, dy + \int_0^x (\pi - x)\cos ry\, dy\right\}$$

$$= \frac{2\cos ry'}{\pi^2}\left\{\frac{\pi \sin r\pi}{r} - \frac{\pi \sin rx}{r} - \Big|_x^\pi \frac{ry \sin ry + \cos ry}{r^2}\right.$$

$$\left. + \frac{(\pi - x)\sin rx}{r}\right\} = \frac{2\cos ry'}{\pi^2}\left\{\frac{\cos rx - \cos r\pi}{r^2}\right\}.$$

$$\frac{4\cos ry' \cos r'x'}{\pi^2}\left\{\int_0^\pi\int_x^\pi(\pi-y) + \int_0^\pi\int_0^x(\pi-x)\right\}\cos ry \cos r'x\, dx\, dy$$

$$= \frac{4\cos ry' \cos r'x'}{\pi^2}\int_0^\pi \frac{(\cos rx - \cos r\pi)\cos r'x}{r^2}\, dx = 0, \text{ if } r \neq r',$$

$$= \frac{4\cos ry' \cos rx'}{\pi^2}\int_0^\pi \frac{dx}{2r^2} = \frac{2\cos ry' \cos rx'}{\pi r^2}, \text{ if } r = r'.$$

Also $\quad \int_0^\pi (\cos rx - \cos r\pi)\, dx = -\pi \cos r\pi.$

We have, therefore,

$$z = \frac{\pi}{3} - \sum_1^s \frac{2\cos r\pi \cos ry'}{\pi r^2} - \sum_1^s \frac{2\cos r\pi \cos rx'}{\pi r^2} + \sum_1^s \frac{2\cos ry' \cos rx'}{\pi r^2}$$

$$= \frac{1}{\pi^2}\left\{(\pi + y')\int_{a(-\pi-x')}^{-ax'} \frac{\sin\phi}{\phi}\,d\phi \int_{a(-\pi-y')}^{a(x'-y')} \frac{\sin\phi}{\phi}\,d\phi\right.$$

$$+ (\pi + y')\int_{-ax'}^{a(\pi-x')} \int_{a(-\pi-y')}^{(a-x'-y')} + (\pi + x')\int_{a(-\pi-x')}^{ax'} \int_{a(x'-y')}^{a(-x'-y')}$$

$$+ (\pi - y')\left(\int_{a(-\pi-x')}^{-ax'} \int_{a(-x'-y')}^{a(\pi-y')} + \int_{-ax'}^{a(\pi-x')} \int_{a(x'-y')}^{a(\pi-y')}\right)$$

$$\left. + (\pi - x')\int_{-ax'}^{a(\pi-x')} \int_{a(-x'-y')}^{a(x'-y')} \right\}.$$

Fig. 8 shews in what compartments each term is effective.

86. As long as we are only concerned with Space and Time, true discontinuity can never arise otherwise than as explained in Chapter I. But in dealing with questions of Natural Philosophy, the question will arise whether the mind is perfectly capable of conceiving a truly discontinuous substance, for instance, a substance the law of whose density is not truly continuous. When we speak of a sphere of uniform density, do we conceive a substance to pervade space which has a zero density outside the sphere, and a constant density, not zero, inside, the passage from one to the other being abrupt? If so, we drop the conception whenever we proceed to reasoning involving the differentiation of such density throughout space, and in its place supply the conception of a mass pervading the universe, in which the density is the same at the same distance r from the centre of the sphere, and has for magnitude a function such as $\Big|^{a=0} \frac{\rho}{\pi}\int_{r-r'}^{r+r'} \frac{\sin a\,dr}{e^r - 2\cos a + e^{-r}}$, where r' is the radius of the sphere, and ρ is the density of a point inside it. Any such function may be chosen, and they may not be equal on the surface of false discontinuity. If inequality there affects in any way the conclusion arrived at by any reasoning, it will be found

that the reasoning assumes a certain character for the function throughout the surface of false discontinuity.

It would seem that, generally, the result of a difference of law, where there is false discontinuity, will shew itself whenever differentiations take place, but will never shew itself whenever integrations take place. This illustrates the statement of Art. 33. In fact we cannot differentiate the density of any conceived substance, without possessing an idea of the substance, quite independent of our experience of the substance, yet so constituted as to chime in with it: being, in fact, one out of an infinite number, which all alike correspond with the experience; that particular one being, in general, selected, which presents itself to the mind as the simplest concept.

CHAPTER III.

OF THE FUNCTIONS WHICH SATISFY DIFFERENTIAL EQUATIONS.

87. In the functions which satisfy differential equations, constants generally occur which do not occur in the original equation. It will be obvious that if such constants be replaced by functions like those spoken of in Arts. 26 to 29, the given differential equation will still be satisfied by the function so altered. Such a solution may be called a discontinuous solution of the differential equation. Thus, for instance, the differential equation, $\frac{dy}{dx} = a$, is not only satisfied by giving to y the value

$$ax + C,$$

where C is constant, but is also satisfied if y have the value expressed by the function

$$y = ax - \frac{c\sqrt{(x-b)^2}}{x-b} - \frac{c\sqrt{(x-2b)^2}}{x-2b} - \ldots\ldots - \frac{c\sqrt{(x-rb)^2}}{x-rb} - \ldots$$

$$- \frac{c\sqrt{(x+b)^2}}{x+b} - \frac{c\sqrt{(x+2b)^2}}{x+2b} - \ldots\ldots - \frac{c\sqrt{(x+rb)^2}}{x+rb} - \ldots$$

This vanishes when $x = 0$, and gives,

when $x = \pm b$, $\quad y = \pm ab \quad$ or $\quad \pm ab \mp 2c$,

... $x = \pm 2b$, $\quad y = \pm 2ab \mp 2c \quad$ or $\quad \pm 2ab \mp 2c \mp 2c$,

and so on; so that by taking $2c = ab$, and b small enough, y may be made to differ from zero by less than any assignable quantity for all values of x, and yet the equation $\frac{dy}{dx} = a$ be still satisfied. In this way, whatever be the value of $\frac{dy}{dx}$, y may be made to differ from any assigned function of x by a quantity less than any assignable quantity. It is obvious that the method may be extended so as to equate to entirely independent functions as many differential coefficients, of + or − orders, as we please. Such solutions may be called truly discontinuous solutions.

88. Now, although it should be recognized that differential equations are satisfied by truly discontinuous functions, yet it ought only to be allowed that such solutions are solutions in the sense that every continuous bit, starting from, and ending with, a point at which true discontinuity occurs, and having none such points between its ends, is a solution of the differential equation, as a differential equation. For, from the nature of a differential coefficient, at a point where true discontinuity occurs, the differential coefficient has only meaning with reference to the direction in which the continuity is unbroken; for every mode of obtaining it, as a differential coefficient, presupposes continuity.

89. But the reasons which lead to disallowing truly discontinuous solutions to be solutions in the same sense as a continuous solution is, do not apply to falsely discontinuous solutions. If an equation exist between x and y, and a function of x, when substituted for y, satisfy the equation, y equal that function is said to be a solution of that equation, as far as that equation is an algebraic or functional equation. In this sense truly discontinuous solutions must be admitted. If further the equation involves in its idea the idea of continuity, as differential equations do, the functions, whose differential coefficients occur in the equation, must not be truly discontinuous. But any function of x, not truly discontinuous between the limits between which the solution is to apply, which, when substituted for y, satisfies the equation by means of itself and its derived functions, should be allowed to be a solution. So falsely discontinuous solutions are as admissible as simpler solutions.

90. If in the equation,

$$\frac{dy}{dx}=f(x),$$

$f(x)$ is always finite, $y=\int f(x)\,dx+C$ is the most general solution consistent with y being not truly discontinuous. For $\frac{dC}{dx}$ must be zero, and the rate of increase of no not truly discontinuous function can be throughout zero, unless that function have always the same value.

But, although if $f(x)$ is ever infinite, and C be a function, expressed by a definite integral (similar in nature to those on which Fourier's Theorem depends, and which we shall represent by $I(x)$,) and $I(x)$ only change value when $f(x)$ is infinite, then

$$y=\int f(x)\,dx+\int I(x)\,dx$$

satisfies the equation $\frac{dy}{dx}=f(x)$. $I(x)$ is a falsely discontinuous function which is zero when $f(x)$ is finite and may be infinite when $f(x)$ is infinite. $\int I(x)\,dx$ is one of those functions which we shall represent by $W(x)$, being a falsely discontinuous function, passing continuously from one constant value to another constant value when $f(x)$ is infinite.

Yet it would be incorrect to regard

$$\eta=\int f(x)\,dx+\int I(x)\,dx$$

as a solution of the equation $\frac{dy}{dx}=f(x)$, notwithstanding that the value of $\frac{d\eta}{dx}$ obtained from it confounds at every point with $f(x)$, for the difference between $\frac{d\eta}{dx}$ and $\frac{dy}{dx}$ is not zero when $f(x)$ is infinite.

So, in every case where $f(x)$ is known in conception as well as in value, $y=\int f(x)\,dx+C$ is the most general solution, consistent with y being not truly discontinuous, of the equation

$$\frac{dy}{dx}=f(x).$$

91. But if we have an equation

$$\left\{\frac{dy}{dx}-f_1(x)\right\}\left\{\frac{dy}{dx}-f_2(x)\right\}=0,$$

$y=\int f_1(x)\,dx+C_1$, $y=\int f_2(x)\,dx+C_2$, are not the only solutions admissible. We may have also solutions wherein $\frac{dy}{dx}$ is truly discontinuous, and y is not truly discontinuous; and, when there are values of x which satisfy $f_1(x)=f_2(x)$, other solutions wherein neither y nor $\frac{dy}{dx}$ are truly discontinuous. Thus, for example, if

$$\left(\frac{dy}{dx}-x\right)\frac{dy}{dx}=0,\quad \text{we can have}$$

$$\pi\frac{dy}{dx}=\left|^{a=0}\; x\int_x^{+\infty}\frac{\sin a\,dz}{e^z-2\cos a+e^{-z}},\right.$$

$$\pi(y-y_0)=\left|^{a=\infty}\;\frac{x^2}{2}\int_x^{+\infty}\frac{\sin a\,dz}{e^z-2\cos a+e^{-z}},\right.$$

in which neither y nor $\frac{dy}{dx}$ are truly discontinuous. We can also have

$$\frac{dy}{dx}=\frac{1}{2}x\left\{\frac{\sqrt{(x-a)^2}}{x-a}+1\right\},$$

$$y-y_0=\frac{x^2-a^2}{4}\left\{\frac{\sqrt{(x-a)^2}}{x-a}+1\right\},$$

in which y is not truly discontinuous, but $\frac{dy}{dx}$ is. (See fig. 9.)

In like manner true and false discontinuity can arise in the solutions of other differential equations of a more complex nature. It should be always carefully remembered that, if a not truly discontinuous function of x, when substituted for y, satisfy a proposed equation between x and y, y equal that function is as much a solution of the proposed equation as any other solution can be.

92. Take the equation

$$y - x\frac{dy}{dx} = \sqrt{1 + \frac{dy^2}{dx}}.$$

Let us investigate whether

$$\pi y = \Big|^{a=0} \Big\{(cx + \sqrt{1+c^2})\int_{c+x\sqrt{1+c^2}}^{+\infty} + \sqrt{1-x^2}\int_{-c_1+x\sqrt{1+c_1^2}}^{c+x\sqrt{1+c^2}}$$

$$+ (-c_1x + \sqrt{1+c_1^2})\int_{-\infty}^{-c_1+x\sqrt{1+c_1^2}}\Big\} \frac{\sin a dz}{e^z - 2\cos\alpha + e^{-z}}$$

be a solution of this equation.

If we differentiate we obtain

$$\pi\frac{dy}{dx} = c\int_{c+x\sqrt{1+c^2}}^{+\infty} - \frac{x}{\sqrt{1-x^2}}\int_{-c_1+x\sqrt{1+c_1^2}}^{c+x\sqrt{1+c^2}} - c_1\int_{-\infty}^{-c_1+\ \sqrt{1+c_1^2}}$$

$$- \frac{\sqrt{1+c^2}\,(cx + \sqrt{1+c^2})\sin\alpha}{e^{c+x\sqrt{1+c^2}} - 2\cos\alpha + e^{-c-x\sqrt{1+c^2}}}$$

$$+ \frac{\sin\alpha\sqrt{1-x^2}\sqrt{1+c^2}}{e^{c+x\sqrt{1+c^2}} - 2\cos\alpha + e^{-c-x\sqrt{1+c^2}}} + \ldots\ldots$$

Retaining only those terms which are of importance when

$$x = -\frac{c}{\sqrt{1+c^2}} + g\alpha,$$

where g is finite, we have

$$\pi\frac{dy}{dx} = c\int_{c+x\sqrt{1+c^2}}^{+\infty} - \frac{x}{\sqrt{1-x^2}}\int_{-c_1+x\sqrt{1+c_1^2}}^{c+x\sqrt{1+c^2}}$$

$$+ \frac{\sin\alpha.\{\sqrt{1+c^2-c^2+2cg\alpha\sqrt{1+c^2}-g^2\alpha^2(1+c^2)}-1-cg\alpha\sqrt{1+c^2}\}}{e^{c+x\sqrt{1+c^2}} - 2\cos\alpha + e^{-c-x\sqrt{1+c^2}}} + \ldots.$$

$$= c\int_{c+x\sqrt{1+c^2}}^{+\infty} - \frac{x}{\sqrt{1-x^2}}\int_{-c_1+x\sqrt{1+c_1^2}}^{c+x\sqrt{1+c^2}} - \frac{R\alpha^2\sin\alpha}{e^{g\alpha\sqrt{1+c^2}} - 2\cos\alpha + e^{-g\alpha\sqrt{1+c^2}}}.$$

So we have

$$\pi\left(y-x\frac{dy}{dx}\right)=\sqrt{1+c^2}\int_{c+x\sqrt{1+c^2}}^{+\infty}+\frac{1}{\sqrt{1-x^2}}\int_{-c_1+x\sqrt{1+c^2}}^{c+x\sqrt{1+c^2}}$$

$$+\sqrt{1+c_1^2}\int_{-\infty}^{-c_1+x\sqrt{1+c^2}};$$

and thence

$$\pi^2\left(y-x\frac{dy}{dx}\right)^2-\pi^2\left(1+\frac{dy^2}{dx}\right)$$

$$=\left(\int_{c+x\sqrt{1+c^2}}^{+\infty}\right)^2+\left(\int_{-c_1+x\sqrt{1+c_1^2}}^{c+x\sqrt{1+c^2}}\right)^2+\left(\int_{-\infty}^{-c_1+x\sqrt{1+c_1^2}}\right)^2$$

$$+2\left(\sqrt{\frac{1+c^2}{1-x^2}}+\frac{cx}{\sqrt{1-x^2}}\right)\int_{c+x\sqrt{1+c^2}}^{+\infty}\int_{-c_1+x\sqrt{1+c^2}}^{c+x\sqrt{1+c^2}}+\ldots\ldots-\pi^2$$

$$=\left(\int+\int+\int\right)^2-\pi^2=0.$$

In like manner it may be shewn that any falsely discontinuous function, which coincides through its whole course with some tangent of the semicircle $y=\sqrt{1-x^2}$, is a solution of the considered differential equation. Any number of constants may be introduced, but it will be observed that only one can be effective at the same time, and if the number of constants exceed two, $\frac{dy}{dx}$ must be truly discontinuous.

93. It is evident that c_1, c_2, may be taken as great as we please in (92). Further no objection can be made to the work of (92) on the ground of $\sqrt{1-x^2}$ being truly discontinuous when $x=1$. For $\Big|^{n=\infty}\,(1-x^2)^{\frac{2n+1}{4n+1}}$ can be substituted for it without affecting the processes. Let us now consider whether the equation

$$\sqrt{a^2-x^2}\,\frac{dy}{dx}=-x$$

admits, when $x=\pm a$, other solution than that furnished by the equation $(y-b)^2+x^2=a^2$.

Put $a^2 = b^2 + \delta^2$, where we shall suppose δ indefinitely small, Consider the function

$$\Big|^{\alpha=0} \pi y = \left\{(cx + b\sqrt{1+c^2}) \int_{cb+x\sqrt{1+c^2}}^{+\infty} + \Big|^{n=\infty} (b^2 - x^2)^{\frac{2n+1}{4n+1}} \int_{-cb+x\sqrt{1+c^2}}^{+cb+x\sqrt{1+c^2}} \right.$$

$$\left. + (-cx + b\sqrt{1+c^2}) \int_{-\infty}^{-cb+x\sqrt{1+c^2}} \right\} \frac{\sin \alpha dz}{e^z - 2\cos\alpha + e^{-z}}.$$

We can connect $\frac{1}{c}$, α, δ, $\frac{1}{n}$, by equations, so that they may bear to each other any ratio we please, and still be all evanescent. When $x = b + g\delta^2$, where $b + g\delta^2 < a$, or $2bg\delta^2 + g^2\delta^4 < \delta^2$, which will be satisfied if g range from zero to $\frac{1}{2b}$, we have

$$\pi y = \left\{- (2bg\delta^2 + g^2\delta^4)^{\frac{2n+1}{4n+1}} \int_{-cb+(b+g\delta^2)\sqrt{1+c^2}}^{+\infty} \right.$$

$$\left. + (-cb - cg\delta^2 + b\sqrt{1+c^2}) \int_{-\infty}^{-cb+(b+g\delta^2)\sqrt{1+c^2}} \right\} \frac{\sin \alpha dz}{e^z - 2\cos\alpha + e^{-z}}$$

$$= -cg\delta^2 \int_{-\infty}^{cg\delta^2} \frac{\sin \alpha dz}{e^z - 2\cos\alpha + e^{-z}}$$

$$= -\pi cg\delta^2, \text{ if } cg\delta^2 \text{ is finite.}$$

Let $OA = a$ (fig. 10) be the radius of the circle ABA', $OT = b$ the radius of a concentric circle. Draw a tangent to the concentric circle meeting the tangent at A in P. We have above an expression for the locus of PTT'. Now, P remaining fixed, let δ diminish till it vanish: the curve PTT' will ultimately coincide with PAB, and furnish a solution to the equation

$$\sqrt{a^2 - x^2}\,\frac{dy}{dx} = -x.$$

It would not be correct to assert that the curve PAB is a solution of the equation $\frac{dy}{dx} = -\frac{x}{\sqrt{a^2 - x^2}}$. Calling y the ordinate of ABA', and η the ordinate of PAB, the value of $(\eta - y)$ when $x = a$

is PA, its general value being zero. Now $\frac{d}{dx}(\eta - y)$ must be infinite when $x = a$, so that $\frac{d\eta}{dx}$ is not $= \frac{dy}{dx}$. At the same time the curve PAB is a solution of the equation $\frac{d\eta}{dx} = -\frac{x}{\sqrt{a^2 - \phi - x^2}}$, where ϕ is a function which is always zero, and

$$\frac{x}{\sqrt{a^2 - \phi - x^2}} - \frac{x}{\sqrt{a^2 - x^2}}$$

is infinite. We shall consider this still further in the next Article.

94. Let $\eta_1 = \frac{x}{\sqrt{a^2 - x^2}}$, $\eta_2 = \frac{x}{\sqrt{a^2 - \phi - x^2}}$, be the equations to two curves, ϕ being a function of x which is always evanescent; the two curves will coincide for all finite values of η_1, but the areas $\int_0^a \eta_1 dx$, $\int_0^a \eta_2 dx$, will not necessarily be equal. Take for ϕ, for example, the function which is zero till $x = b$, and then has for value $(a^2 - x^2)\left(1 - \frac{a^2 - x^2}{a^2 - b^2}\right)$ till $x = a$, and is afterwards zero. Such a function is the expression

$$\frac{(a^2 - x^2)(x^2 - b^2)}{\pi (a^2 - b^2)} \Bigg|^{\alpha = 0} \int_{x-a}^{x-b} \frac{\sin adz}{e^z - 2\cos\alpha + e^{-z}},$$

even when a becomes $= b$, provided $\frac{\alpha}{a - b}$ still vanish (Art. 73).

Now, $$\int_0^a = \int_0^b + \int_b^a, \text{ and}$$

$$\int_0^b \frac{xdx}{\sqrt{a^2 - \phi - x^2}} = \int_0^b \frac{xdx}{\sqrt{a^2 - x^2}},$$

$$\int_b^x \frac{xdx}{\sqrt{a^2 - \phi - x^2}} = \int_b^x \frac{xdx\sqrt{a^2 - b^2}}{a^2 - x^2}$$

$$= -\frac{1}{2}\sqrt{a^2 - b^2} \Big|_b^x \log(a^2 - x^2)$$

$$= \frac{1}{2}\sqrt{a^2 - b^2} \log\frac{a^2 - b^2}{a^2 - x^2},$$

where x is any quantity between b and a.

Let $a = b + \delta$, then

$$\int_b^{a-\delta^{\frac{1}{\delta}}} \frac{xdx}{\sqrt{a^2 - \phi - x^2}} = \frac{1}{2}\sqrt{2b\delta + \delta^2} \log \frac{2b\delta + \delta^2}{2a\delta^{\frac{1}{\delta}} - \delta^{\frac{2}{\delta}}}$$

$$= -\sqrt{\frac{b}{2\delta}} \log \delta \text{ ultimately,}$$

which is infinite.

Or we may proceed thus:

$$\int_b^{a-\delta_1} \frac{xdx}{\sqrt{a^2 - \phi - x^2}} = \frac{1}{2}\sqrt{2b\delta + \delta^2} \log \frac{2b\delta + \delta^2}{2a\delta_1 - \delta_1^2}$$

$$= \frac{1}{2} \log (2b\delta + \delta^2)^{\sqrt{2b\delta+\delta^2}} + \frac{1}{2} \log (2a\delta_1 - \delta_1^2)^{-\sqrt{2b\delta+\delta^2}};$$

determine δ_1 from the equation

$$\delta_1^{-\sqrt{\delta}} = g, \quad \text{or } \delta_1 = g^{-\frac{1}{\sqrt{\delta}}},$$

and we have a finite value for $\int_b^{a-\delta_1} \frac{xdx}{\sqrt{a^2 - \phi - x^2}}$, when δ, and therefore δ_1, vanishes.

95. Be it observed that there is nothing in the two preceding Articles which militates against the use of the symbol $\int_a^b f'(x)dx$, even when $f'(x)$ becomes infinite between the limits, provided $f(x)$ remain finite throughout, when it always properly represents $f(b) - f(a)$. But if our knowledge of a function is limited to this, that we only know that it coincides in value with another function whensoever that function is finite, then we are not able to assert that its integral coincides in value with the integral of the other function, when the infinite value occurs between the limits between which the integration is taken.

We shall find that functions like

$$\Big|^{\alpha=0} \phi(x) \int_x^{x+\delta} \frac{\sin \alpha dz}{e^z - 2 \cos \alpha + e^{-z}},$$

where $\frac{a}{\delta}$ is zero, are of frequent occurrence in problems solved by Calculus of Variations, as we shall solve them. A not truly discontinuous function which is generally zero, but for a particular value is not so, will be represented by $I(x)$. It is evident that $\int_a^b I(x)\,dx = 0$, when $I(x)$ is never infinite. Also $W(x)$ will represent not truly discontinuous functions generally constant, but having different values. It is clear that $\frac{dW(x)}{dx} = I(x)$, where $I(x)$ is sometimes infinite.

96. Suppose that it is required to find what function y is of x, when, b^2 being either + or −,

$$(y^2 - b^2)\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 2ay.$$

If a conic section, whose transverse axis is $2a$, and eccentricity $\frac{\sqrt{a^2 + b^2}}{a}$, roll along the axis of x, its focus will trace out a curve at every point of which the relation

$$\left(\frac{dy}{dx}\right)^2 = \frac{4a^2y^2 - (y^2 - b^2)^2}{(y^2 - b^2)^2}$$

is satisfied. (Jellett, p. 364, he ascribes the construction to Delaunay.)

The constructing conic section will be a hyperbola if $\frac{dy}{dx}$ be ever infinite, which will occur when b^2 is intermediate to

$$(\sqrt{a^2 + b^2} + a)^2 \text{ and } (\sqrt{a^2 + b^2} - a)^2,$$

i.e., when b^2 is +, a parabola when a is infinite, and $\frac{b^2}{a}$ is finite, and an ellipse when $\frac{dy}{dx}$ is always real and finite. When b^2 vanishes, the conic section becomes a terminated straight line when a is finite, and the curve traced a succession of semicircles.

When the describing conic is a hyperbola, (as $\frac{y}{y^2-b^2}$ must be $+$,) the rolling hyperbola must turn round the ordinate through the focus, when its asymptote has rolled into coincidence with the axis of x, before rolling further (fig. 11).

Thus, if a hyperbola roll on the under side of the axis of x, so that the nondescribing focus describe the curve $H_0H_1H_2$, the describing focus will trace the curve $S_0S_1S_2$; and, before it proceed to describe another period, the hyperbola must revolve round the ordinate NS_2, till its focus come to H_2'. Otherwise, (*i.e.* if the other branch of the hyperbola roll on the axis of x,) we shall have a curve wherein $\frac{y}{y^2-b^2}$ is $-$.

Now, if at any point in the periodic curve $S_0S_1S_2S_4$, $\frac{dy}{dx}$ change sign, our differential equation will still be satisfied. Also if, whenever y^2 has its maximum value, the ordinate retain a constancy value for any distance.

Thus in the curve (fig. 12), wherein y is constant from S_3 to S_4, and $\frac{dy}{dx}$ abruptly (being truly discontinuous) changes sign $\left(\frac{dy^2}{dx}\text{ and } y \text{ being falsely discontinuous}\right)$ at S_5, S_8, S_9, ... the relation

$$\sqrt{\frac{dy^2}{dx}+1}=\frac{2ay}{y^2-b^2}$$

is satisfied.

Now, by Fourier's theorem, we can obtain an expression,

$$\psi(x+\xi)=A_1'\sin\frac{2(x+\xi)\pi}{l}+A_2'\sin\frac{4(x+\xi)\pi}{l}+\dots$$
$$+A_r'\sin\frac{2r(x+\xi)\pi}{l}+\dots,$$

which shall coincide in value with the value of

$$\left\{(y^2-b^2)\frac{dy}{dx}\right\}^2,$$

throughout the curve $S_0S_1S_2$, and also an expression for y of the form

$$\phi(x+\xi) = A_1 \cos\frac{2(x+\xi)\pi}{l} + A_2 \cos\frac{4(x+\xi)\pi}{l} + \dots$$

wherein all the coefficients A_1', A_2', ... A_1, A_2, are finite. This being done the expression for $\left\{(y^2-b^2)\frac{dy}{dx}\right\}^2$ in the curve $S_0S_2 \dots S_9$ will be of the form

$$\frac{1}{\pi}\left\{+\psi(x+\xi_1)\int_{x_2}^{x_1} + \psi(x+\xi_2)\int_{x_3}^{x_2} + 0\int_{x_4}^{x_3} + \psi(x+\xi_4)\int_{x_5}^{x_4}\right.$$

$$\left. + \psi(x+\xi_5)\int_{x_6}^{x_5} + 0\int_{x_7}^{x_6} + \dots\right\}\frac{\sin 0\, dz}{e^z - 2\cos 0 + e^{-z}},$$

and similarly for y. It is to be understood that ξ_1, ξ_2, ... must be so chosen that $\left(\frac{dy}{dx}\right)^2(y^2-b^2)^2$ is not truly discontinuous.

Similar considerations apply to the loci wherein

$$-(y^2-b^2)\sqrt{1+\left(\frac{dy}{dx}\right)^2} = 2ay$$

is satisfied.

In the particular case of $b^2 = 0$, not only the constancy value $2a$, but also the constancy value 0, for y, will satisfy

$$y^2\sqrt{1+\left(\frac{dy}{dx}\right)^2} = 2ay.$$

So that the curve, (consisting of arcs of circles with centre on the axis of x and radius $2a$, when $y^2 \neq 2ay$,) (fig. 13) is one solution, and its equation can be, in like manner, expressed.

The expression for y in terms of s is simpler than that in terms of x. For the simplest locus it is, within limits for s,

$$y^2 = 2a^2 + b^2 - 2a\sqrt{a^2+b^2}\cos\frac{s-\sigma}{a}.$$

97. Supposing b^2+, and no true discontinuity in $\frac{dy}{dx}$, the greatest distance between successive cusps, obtained by varying b, in the curves of (96) will be $4a$, and the least distance $2a$. Also the problem of drawing such a curve so as to pass through two given points often admits of more than one solution (fig. 14).

Select the coordinate axes so that x vanishes when $\frac{dy}{dx}$ vanishes. Then the curves will be symmetrical with respect to the axis of y, and

$$\pm x = \int_y^{a+\sqrt{a^2+b^2}} \frac{y^2-b^2}{\sqrt{4a^2y^2-(y^2-b^2)^2}}\,dy.$$

So that, the distance between the cusps being ξ,

$$\xi = 2\int_b^{a+\sqrt{a^2+b^2}} \frac{y^2-b^2}{\sqrt{4a^2y^2-(y^2-b^2)^2}}\,dy.$$

Let $\pm\frac{dy}{dx} = -\cot\phi$, so that $y^2-b^2 = 2ay\sin\phi$,

$$(y-a\sin\phi)\frac{dy}{d\phi} - ay\cos\phi = 0,$$

$$\xi = 2\int_0^{\frac{\pi}{2}} \frac{ay\sin\phi\,.\,d\phi}{\sqrt{a^2\sin^2\phi+b^2}} = 2\int_0^{\frac{\pi}{2}} \frac{a^2\sin^2\phi\,d\phi}{\sqrt{a^2\sin^2\phi+b^2}} + 2a.$$

The least value of this is $2a$, and the greatest $4a$. Also ξ decreases as b^2 increases, for

$$\frac{d\xi}{db} = -2\int_0^{\frac{\pi}{2}} \frac{a^2b\sin^2\phi\,d\phi}{(a^2\sin^2\phi+b^2)^{\frac{3}{2}}},$$

which is $-$ and finite when b is finite.

It follows that the curve which is the locus of the cusps cuts each curve in more than one point, as is represented in fig. 14. So that if any curve of the series be drawn, other than that for which $b=0$, another curve of the series can be drawn through its cusps. The radius of curvature throughout is $\frac{2ay^2}{y^2+b^2}$, and so the semicircle, obtained when $b = 0$, has the peculiarity of a falsely discontinuous radius of curvature, its value being $2a$ at points finitely distant from Ox, and $\frac{2ac}{1+c}$ when $y^2 = cb^2$.

98. The nature of the curve of cusps in (fig. 14) is such that ξ cannot be expanded in a series of powers of b. For

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a^2 \sin^2 \phi + b^2}} > \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a^2 \phi^2 + b^2}},$$

therefore,
$$> \frac{1}{a} \log \left(\frac{\pi}{2} + \sqrt{\frac{\pi^2}{4} + \frac{b^2}{a^2}}\right) - \frac{1}{a} \log \frac{b}{a}.$$

And
$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a^2 \sin^2 \phi + b^2}} < \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a^2 \sin^2 \phi + b^2}} + \int_0^{\frac{\pi}{4}} \frac{\sec^2 \phi \, d\phi}{\sqrt{(a^2 - b^2) \tan^2 \phi + b^2}},$$

therefore,
$$< \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a^2 \sin^2 \phi + b^2}}$$

$$+ \frac{1}{\sqrt{a^2 - b^2}} \left\{ \log \left(1 + \sqrt{1 + \frac{b^2}{a^2 - b^2}}\right) - \log \frac{b}{\sqrt{a^2 - b^2}} \right\}.$$

So that, ultimately,

$$\Bigg|^{b^2=0} \frac{-1}{\log b} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a^2 \sin^2 \phi + b^2}} = \frac{1}{a}.$$

Again,
$$\int_0^{\frac{\pi}{2}} \left(\sqrt{a^2 \sin^2 \phi + b^2} - a \sin \phi\right) d\phi$$

$$= \int_0^{\frac{\pi}{2}} \frac{b^2 \, d\phi}{\sqrt{a^2 \sin^2 \phi + b^2} + a \sin \phi}$$

$$> \int_0^{\frac{\pi}{2}} \frac{b^2 \, d\phi}{2\sqrt{a^2 \sin^2 \phi + \frac{b^2}{2}}} < \int_0^{\frac{\pi}{2}} \frac{b^2 \, d\phi}{2\sqrt{a^2 \sin^2 \phi + \frac{b^2}{4}}}.$$

For $2a^2 \sin^2\phi + b^2 + 2a \sin\phi \sqrt{a^2 \sin^2\phi + b^2} < 4a^2 \sin^2\phi + 2b^2$,
and is $> 4a^2 \sin^2\phi + b^2$.

So that, ultimately,

$$\frac{2\int_0^{\frac{\pi}{2}} (\sqrt{a^2 \sin^2\phi + b^2})\, d\phi - 2a}{-b^2 \log b}$$

$$> \frac{-1}{\log b} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a^2 \sin^2\phi + \frac{b^2}{2}}} < \frac{-1}{\log b} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a^2 \sin^2\phi + \frac{b^2}{4}}},$$

and, therefore, ultimately $= \frac{1}{a}$.

So that $\qquad \left|^{b^2=0} \frac{\xi - 4a}{-b^2 \log b} = \frac{1}{a} - \frac{2}{a} = -\frac{1}{a}. \right.$

99. In (fig. 14) another curve is drawn more remote from the line, $x = 0$, than the line of cusps. It represents the locus of the ultimate intersections, obtained by varying b, of the curves

$$-x = \int_y^{a+\sqrt{a^2+b^2}} \frac{(y^2 - b^2)\, dy}{\sqrt{4a^2y^2 - (y^2 - b^2)^2}}.$$

Now (97) the equation to this locus may be written

$$-x = \int_\phi^{\frac{\pi}{2}} \frac{ay \sin\phi\, d\phi}{\sqrt{a^2 \sin^2\phi + b^2}} = \int_\phi^{\frac{\pi}{2}} \frac{a^2 \sin^2\phi\, d\phi}{\sqrt{a^2 \sin^2\phi + b^2}} + a\cos\phi,$$

giving

$$-\frac{dx}{db} = -\int_\phi^{\frac{\pi}{2}} \frac{a^2 b \sin^2\phi\, d\phi}{(a^2 \sin^2\phi + b^2)^{\frac{3}{2}}} - \left\{\frac{a^2 \sin^2\phi}{\sqrt{a^2 \sin^2\phi + b^2}} + a \sin\phi\right\} \frac{-b}{ay\cos\phi}.$$

For $\qquad y^2 - b^2 = 2ay \sin\phi,$

$$\therefore -\frac{dx}{db} = -\int_\phi^{\frac{\pi}{2}} \frac{a^2 b \sin^2\phi\, d\phi}{(a^2 \sin^2\phi + b^2)^{\frac{3}{2}}} + \frac{b \tan\phi}{\sqrt{a^2 \sin^2\phi + b^2}},$$

and the elimination of b, ϕ between

$$0 = -\int_\phi^{\frac{\pi}{2}} \frac{a^2 b \sin^2\psi\, d\psi}{(a^2\sin^2\psi + b^2)^{\frac{3}{2}}} + \frac{b\tan\phi}{\sqrt{a^2\sin^2\phi + b^2}},$$

$$y^2 - b^2 = 2ay\sin\phi,$$

$$\frac{dy}{dx} = \cot\phi,$$

will give the locus of the ultimate intersections. The above integral can be reduced to a more recognizable form by observing that

$$\frac{d}{d\phi}\left(\frac{\sin\phi\cos\phi}{\sqrt{a^2\sin^2\phi + b^2}}\right) - \frac{\cos^2\phi}{\sqrt{a^2\sin^2\phi + b^2}}$$

$$= \frac{(a^2\sin^2\phi + b^2)(\cos^2\phi - \sin^2\phi - \cos^2\phi) - a^2\sin^2\phi\cos^2\phi}{(a^2\sin^2\phi + b^2)^{\frac{3}{2}}}$$

$$= -\frac{a^2\sin^2\phi + b^2\sin^2\phi}{(a^2\sin^2\phi + b^2)^{\frac{3}{2}}}.$$

It must not be supposed that this locus will satisfy the differential equation obtained by differentiating, b being constant,

$$b^2 = y^2 - \frac{2ay}{\sqrt{1 + \frac{dy^2}{dx}}},$$

because it satisfies a first integral of that equation, viz.:

$$-x + 0 = \int_y^{a + \sqrt{a^2 + y^2 - \frac{2ay}{\sqrt{1+p^2}}}} \frac{z^2 - y^2 + \frac{2ay}{\sqrt{1+p^2}}}{\sqrt{4a^2z^2 - \left(z^2 - y^2 + \frac{2ay}{\sqrt{1+p^2}}\right)^2}}\, dz.$$

It will be found on investigation that the differential equation obtained from this last will assume the form

$$M\left\{\left(2y - \frac{2a}{\sqrt{1+p^2}}\right)\frac{dy}{dx} + \frac{2apy}{(1+p^2)^{\frac{3}{2}}}\frac{dp}{dx}\right\} + \frac{1}{p}\frac{dy}{dx} - 1 = 0,$$

where $\dfrac{dy}{dx} = p$, and the locus of the ultimate intersections which we have investigated will satisfy $M = 0$.

100. When b^2 is $-$, the constructing conic is an ellipse; the curve traced by its focus will be concave towards the axis of x when $y^2 > -b^2$, and convex when $y^2 < -b^2$. If we select the coordinate axes so that x vanishes when $\dfrac{dy}{dx}$ vanishes, we can find, as in Art. 97, the locus of the points for which $\dfrac{d^2y}{dx^2}$ vanishes, $-b^2$ being supposed to vary. In fact its equation will be

$$\pm x = \int_y^{a+\sqrt{a^2-y^2}} \frac{z^2+y^2}{\sqrt{4a^2z^2-(z^2+y^2)^2}}\,dz.$$

To simplify this, assume

$$z^2 - y^2 = 2z\sqrt{a^2-y^2}\sin\psi.$$

When $\quad z = a + \sqrt{a^2-y^2}, \quad \sin\psi = 1; \quad$ when $z = y, \quad \sin\psi = 0;$

and $\quad (z - \sqrt{a^2-y^2}\sin\psi)\dfrac{dz}{d\psi} = z\sqrt{a^2-y^2}\cos\psi,$

$$z - \sqrt{a^2-y^2}\sin\psi = \sqrt{a^2\sin^2\psi + y^2\cos^2\psi},$$

$$(z^2+y^2)^2 = 4z^2(a^2-y^2)\sin^2\psi + 4y^2z^2 = 4z^2(a^2\sin^2\psi + y^2\cos^2\psi).$$

So that

$$\pm x = \int_0^{\frac{\pi}{2}} \frac{2z\sqrt{a^2\sin^2\psi+y^2\cos^2\psi}}{2z\sqrt{(a^2-y^2)\cos^2\psi}} \cdot \frac{z\sqrt{a^2-y^2}\cos\psi}{\sqrt{a^2\sin^2\psi+y^2\cos^2\psi}}\,d\psi$$

$$= \int_0^{\frac{\pi}{2}} z\,d\psi = \int_0^{\frac{\pi}{2}} \sqrt{a^2\sin^2\psi + y^2\cos^2\psi}\,d\psi + \sqrt{a^2-y^2}.$$

This will give

$$\pm\frac{dx}{dy} = \int_0^{\frac{\pi}{2}} \frac{y\cos^2\psi\,d\psi}{\sqrt{a^2\sin^2\psi+y^2\cos^2\psi}} - \frac{y}{\sqrt{a^2-y^2}}.$$

Now if we trace, for a particular value of β^2, the locus $\left(\frac{dy}{dx}\right.$ vanishing when x vanishes$\left.\right)$, (fig. 15),

$$\frac{dy^2}{dx}+1=\frac{4a^2y^2}{(y^2+\beta^2)^2},$$

since when $\qquad y=\beta,\ \frac{dy}{dx}=-\sqrt{\frac{a^2-\beta^2}{\beta^2}},$

for y diminishes as x increases, x being positive.

So $\left(\frac{dx}{dy}\right)^2$ is less in the locus of the points for which $\frac{d^2y}{dx^2}$ vanishes than it is in the curve meeting it. And, for the same value of y, an increase in β diminishes $\left(\frac{dy}{dx}\right)^2$. So consecutive curves of the series,

$$\frac{dy^2}{dx}+1=\frac{4a^2y^2}{(y^2+\beta^2)^2},$$

intersect for some value of x between zero and

$$\pm\int_{\beta}^{a+\sqrt{a^2-\beta^2}}\frac{y^2+\beta^2}{\sqrt{4a^2y^2-(y^2+\beta^2)^2}}\,dy.$$

So they will lie as represented in (fig. 15), where the locus of ultimate intersections, and the locus of the points at which $\frac{d^2y}{dx^2}$ vanishes, are both drawn.

The reader will find no difficulty in establishing, for the case of b^2 −, propositions similar to those which we have, in Articles 97 to 99, established for the case of b^2 +.

This subject will be further considered in the Appendix.

CHAPTER IV.

OF MAXIMA AND MINIMA.

101. If a real value can be assigned to r such that, although $f(x'+r)$ is not equal to $f(x')$, yet $f(x'+r\cos\theta)-f(x')$ has the same sign for all real values of θ, then $f(x')$ is said to be a maximum when that sign is negative, and a minimum when that sign is positive; and $f(x)$ is said to have a maximum or a minimum value, according as that sign is negative or positive, when $x=x'$.

It should be observed that, if $f(x'+r\cos\theta)$ is imaginary, or constant in value, for some non-evanescent value of r, $f(x')$ being real, then $f(x')$ may be considered to be both a maximum and a minimum. For distinction we shall call $f(x')$ an isolate, or a conjugate, value of $f(x)$, when $f(x'+r\cos\theta)$ is imaginary, and we shall call $f(x')$ a constancy value when $f(x'+r\cos\theta)-f(x')$ vanishes; r being some non-evanescent real ratio, and θ having any real value.

102. In like manner if a real value can be so assigned to r that, although $f(x_1'+r, x_2'+r, \ldots\ldots x_n'+r)$ is not equal to $f(x_1', x_2', \ldots x_n')$, yet $f(x_1'+r\cos\theta_1, x_2'+r\cos\theta_2, \ldots x_n'+r\cos\theta_n)$ $-f(x_1', x_2', \ldots\ldots x_n')$ has the same sign for all real values of $\theta_1, \theta_2, \ldots \theta_n$, then $f(x_1', x_2', \ldots x_n')$ is said to be a maximum when that sign is $-$, and a minimum when that sign is $+$; and $f(x_1, x_2, \ldots x_n)$ is said to have a maximum or a minimum value, according as that sign is $-$ or $+$, when $x_1=x_1'$, $x_2=x_2'$, ... $x_n=x_n'$.

These functions can have isolate values, or constancy values, in like manner precisely as functions of one independent variable can.

103. If, for a particular value of x, $f(x)$ is truly discontinuous, its then value may be a maximum, or a minimum, or both. Such values should be always separately examined, to see what their nature is.

But, whenever $f(x)$ is not truly discontinuous, the equation in δx,

$$f(x+\delta x)-f(x)=\delta f,$$

will have among its roots a root which, as δf continuously changes from zero, will continuously change from zero, always for one sign of δf, and generally whatever sign δf have. If there is no such real root when δf is negative, $f(x)$ will have a minimum value, and if there is no such real root when δf is positive, $f(x)$ will have a maximum value.

104. When, at a particular value of x, $f(x)$ is not truly discontinuous, both $\{f(x+\delta x)-f(x)\}$ and $\{f(x-\delta x)-f(x)\}$ continuously change from zero as δx continuously changes from zero. The test of Art. 103 requires that they both have the same sign initially when $f(x)$ is a maximum or a minimum. Now the initial value of $\dfrac{f(x)-f(x-\delta x)}{\delta x}=f'(x-\delta x)$, and the initial value of $\dfrac{f(x+\delta x-\delta x)-f(x+\delta x)}{-\delta x}=f'(x+\delta x)$.

So that $f'(x-\delta x)$ and $f'(x+\delta x)$ have different signs initially, changing from + to − for a maximum, and from − to + for a minimum. This cannot take place unless $f'(x)$ is zero, or truly discontinuous and changing sign, for that particular value of x.

105. Whenever $f(x, y)$ is not truly discontinuous, the simultaneous equations in δx, δy,

$$f(x+\delta x,\ y+\delta y)-f(x,\ y)=\delta f,$$

$$\delta x-k\delta y=0,$$

will have, among the systems of real roots which solve them, a system such that, as δf continuously changes from zero, both δx and δy continuously change from zero, whatever positive or negative ratio k be, always for one sign of δf, and generally whatever sign δf have. If there is no such system when δf is positive, $f(x, y)$ will have a maximum value. If there is no such system when δf is negative, $f(x, y)$ will have a minimum value.

Now, whatever value k have, the initial value of

$$\frac{f(x,\ y+\delta y)-f(x-\delta x,\ y+\delta y)}{\delta x}=\Bigg|^{x=x-\delta x}_{y=y+\delta y}\frac{df(x,\ y)}{dx},$$

and the initial value of

$$\frac{f(x+\delta x-\delta x,\ y+\delta y)-f(x+\delta x,\ y+\delta y)}{-\delta x}=\Bigg|^{x=x+\delta x}_{y=y+\delta y}\frac{df(x,\ y)}{dx}.$$

So that $\frac{df}{dx}$ changes sign, from + to − for a maximum, and from − to + for a minimum. It must, therefore, be zero, or truly discontinuous and changing sign, for these particular simultaneous values of x and y. In the same way it can be shown that $\frac{df}{dy}$ changes sign, and is, therefore, zero, or truly discontinuous and changing sign. These conditions require that, and are satisfied if, for all values of k,

$$\frac{df}{dx}\cdot\frac{\delta x}{\sqrt{\delta x^2+\delta y^2}}+\frac{df}{dy}\cdot\frac{\delta y}{\sqrt{\delta x^2+\delta y^2}},\quad\text{or}\quad\frac{k\frac{df}{dx}+\frac{df}{dy}}{\sqrt{1+k^2}}$$

change sign as x, y, pass continuously through their particular simultaneous values.

106. When $f(x, y)$ is truly discontinuous, there are, in general, signs for δx and δy, and limits between which k lying, the simultaneous equations in δx, δy,

$$f(x+\delta x,\ y+\delta y)-f(x,y)=\delta f,$$

$$\delta x-k\delta y=0,$$

have among the systems of real roots which solve them a system such that, as δf continuously changes from zero, both δx and δy continuously change from zero. For these signs and limits the method of Art. 105 must be satisfied, and in every individual case we must examine whether for other signs of δx, δy, and other values of k, we can obtain initially

$$\{f(x+\delta x,\ y+\delta y)-f(x,\ y)\}$$

positive, or negative, or both.

107. It will be readily observed that, for any number of independent variables the plan pursued in Art. 105 will give similar tests for maxima and minima. If there are $(n+1)$ independent variables, we have in general to investigate the existence of a system of roots continuous through zero for the $(n+1)$ simultaneous equations in δx, δy, ... δz,

$$\begin{aligned} f(x+\delta x,\ y+\delta y, \dots z+\delta z)-f(x,\ y, \dots z) &= \delta f, \\ \delta x + k_1 \delta y + \dots\dots + l_1 \delta z &= 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots &, \\ \delta x + k_n \delta y + \dots\dots + l_n \delta z &= 0, \end{aligned}$$

and shall find in like manner that

$$\frac{1}{\sqrt{1+\mu_1^2+\dots+\mu_n^2}}\left(\frac{df}{dx}+\mu_1\frac{df}{dy}+\dots+\mu_n\frac{df}{dz}\right)$$

changes sign as $x, y, \dots z$ pass continuously through their particular simultaneous values. It is obvious that

$$\begin{vmatrix} 1, \dots\dots l_1 \\ 1, \dots\dots l_2 \\ \dots\dots\dots\dots \\ 1, \dots\dots l_n \end{vmatrix} + \mu_1 \begin{vmatrix} k_1, \dots\dots l_1 \\ k_2, \dots\dots l_2 \\ \dots\dots\dots\dots \\ k_n, \dots\dots l_n \end{vmatrix} = 0, \text{ \&c.}$$

108. We will now give illustrations of Arts. 101 to 107.

Ex. (Todhunter, *Differential Calculus*, XIII. Ex. 27.) "A person being in a boat b miles from the nearest point of the beach, wishes to reach in the shortest time a place c miles from that point along the shore; supposing he can walk u miles an hour, but pull only at the rate of v miles an hour, where $v < u$, required the place where he must land."

$AB = b$, $AQ = c$, $AP = x$, where P is the place where he must land (fig. 16). The time of journey must be a minimum. Now

$$\text{time of journey} = t = \frac{\sqrt{b^2 + x^2}}{v} + \frac{\sqrt{(c-x)^2}}{u},$$

$$\frac{dt}{dx} = \frac{x}{v\sqrt{b^2 + x^2}} - \frac{c-x}{u\sqrt{(c-x)^2}},$$

and is truly discontinuous when $x = c$, and may vanish when

$$u^2x^2 = v^2(b^2 + x^2),$$

or

$$x^2 = \frac{b^2v^2}{u^2 - v^2}.$$

If c is $+$, and $< \dfrac{bv}{\sqrt{u^2 - v^2}}$, $\dfrac{dt}{dx}$ is $-$ from $x = -\infty$ till $x = c$, when it changes from $\left(\dfrac{c}{v\sqrt{b^2 + c^2}} - \dfrac{1}{u}\right)$ to $\left(\dfrac{c}{v\sqrt{b^2 + c^2}} + \dfrac{1}{u}\right)$ abruptly. That is, $x = c$ makes t a minimum.

In this case he must land at Q.

If c is $> \dfrac{bv}{\sqrt{u^2 - v^2}}$, $\dfrac{dt}{dx}$ is $-$ from $x = -\infty$ till $x = \dfrac{bv}{\sqrt{u^2 - v^2}}$, when it vanishes and changes sign, never changing sign again. For when $x = c$ it changes abruptly from one $+$ value to another $+$ value.

In this case he must land at P, where $AP = \dfrac{bv}{\sqrt{u^2 - v^2}}$.

The case of $c -$ need not be considered.

109. Ex. To find when y is a maximum or a minimum, when

$$y = (\cos x + \sqrt{-1}\sin x)^{\frac{1}{n}} + (\cos x - \sqrt{-1}\sin x)^{\frac{1}{n}}.$$

We will confine ourselves to the case when n is odd.

We will first say a few words as to the meaning of the above expression considered as expressing a function (Arts. 6, 7).

All the real roots of the equation $x^n = a^2$, being determinate when n is specified, are functions of n and a: they are two in number, one + and one −, when n is even, and one in number, which is +, when n is odd. Consequently only one real root of the equation $x^n = a^2$ is an interpolation between real roots of the equations $x^{n+1} = a^2$, and $x^{n-1} = a^2$. So this real root is more worthy of having a symbol assigned to express it than any other of the roots. In accordance with usage we represent it by $\sqrt[n]{a^2}$, or $(a^2)^{\frac{1}{n}}$, bearing in mind that the latter symbol could not, without an abuse of symbols, be employed otherwise than to express a function which satisfies the equation $x^n = a^2$.

From this we readily obtain the meaning of the function $(a^2)^b$, whatever ratio b be. $(a^2)^{\frac{m}{n}}$ is the m^{th} power of $(a^2)^{\frac{1}{n}}$, and $(a^2)^b$, when b is incommensurable, has the meaning which interpolation assigns to it.

So that $1^b = 1$, whatever real ratio b be.

Again, the equation $x^n = -a^2$ has no real root when n is even, and only one, which is −, when n is odd. The symbol $(-a^2)^{\frac{1}{n}}$ could not, without an abuse of symbols, be employed otherwise than to express a function which satisfies the equation, $x^n = -a^2$. Similarly $-(-a^2)^{\frac{1}{n}}$ should express a root of the equation,

$$x^n = -(-1)^n a^2.$$

Now the imaginary roots enter an equation, like $x^n = -a^2$, by pairs, and no symbol corresponds to $(-a^2)^{\frac{1}{n}}$, when n is odd, so as manifestly to be conjugate to it, and also satisfy the equation, $x^n = -a^2$. Also a real root of an equation is more worthy of having a symbol assigned to express it than any of the imaginary roots. For these reasons $(-a^2)^{\frac{1}{n}}$ is conventionally used, when n is odd, to represent the real root of the equation, $x^n = -a^2$.

And, when n is even, both $(-a^2)^{\frac{1}{n}}$ and $-(-a^2)^{\frac{1}{n}}$ should represent a root. Let $n = 2^t m$, where m is odd. Then should

$$\{-(-a^2)^{\frac{1}{n}}\}^{2^t} = \{+(-a^2)^{\frac{1}{n}}\}^{2^t} = \{(-a^2)^{\frac{2}{n}}\}^{2^{t-1}} = (-a^2)^{\frac{1}{m}}.$$

And this will always enable us to determine which two roots shall be represented by $+(-a^2)^{\frac{1}{n}}$ and $-(-a^2)^{\frac{1}{n}}$, but which will represent which of the two will still be arbitrary, but practically settled by a convention, to which we are led by the principle of simplicity. This convention decides that

$$+(-1)^{\frac{1}{4}} = \cos\frac{\pi}{4} + \sqrt{-1}\sin\frac{\pi}{4},$$

$$+(-1)^{\frac{1}{2^n}} = \cos\frac{\pi}{2^n} + \sqrt{-1}\sin\frac{\pi}{2^n},$$

these roots having a more positive appearance than any of the other roots of $x^4 = -1$, or $x^{2^n} = -1$.

Bearing, then, in mind that if equals be raised to equal integral powers the results are equal, it is easy to see that $(-1)^{\frac{1}{n}}$, and also $(-1)^{\frac{m}{n}}$, can be expressed always in the form $(-1)^{\frac{s}{2^t}}$. Thus for instance

$$(-1)^{\frac{2r}{2s+1}} = 1,\ (-1)^{\frac{2r+1}{2s+1}} = -1,$$

$(-1)^{\frac{2r+1}{2^t(2s+1)}}$ is a root of the equation $x^{2^t} = -1$, all whose roots are contained in the forms $(-1)^{\frac{1}{2^t}}$, $(-1)^{\frac{3}{2^t}}$, ...$(-1)^{\frac{2^t-1}{2^t}}$, and the same forms with their signs changed, so that there is some integer, m, such that

$$(-1)^{\frac{2r+1}{2^t(2s+1)}} = \pm(-1)^{\frac{2m+1}{2^t}} = (-1)^{\frac{2m+1}{2^t}},$$

according as m is restricted to be $< 2^{t-1}$ or $< 2^t$.

So also $+(+\sqrt{-1})^{\frac{2r}{2s+1}} = (-1)^{\frac{r}{2s+1}} = (-1)^r = +(-\sqrt{-1})^{\frac{2r}{2s+1}}$;

$$+(+\sqrt{-1})^{\frac{2r+1}{2s+1}} = +(-1)^{\frac{2r+1}{2(2s+1)}};\quad +(-\sqrt{-1})^{\frac{2r+1}{2s+1}} = -(-1)^{\frac{2r+1}{2(2s+1)}};$$

$$+(+\sqrt{-1})^{\frac{2r+1}{2^t(2s+1)}} = +(-1)^{\frac{2r+1}{2^{t+1}(2s+1)}};$$

$$+(-\sqrt{-1})^{\frac{2r+1}{2^t(2s+1)}} = +\{(-1)^{\frac{3}{2}}\}^{\frac{2r+1}{2^t(2s+1)}} = +(-1)^{\frac{6r+3}{2^{t+1}(2s+1)}};$$

$$+(+\sqrt{-1})^{\frac{7}{12}}=(-1)^{\frac{7}{24}}=\{(-1)^{\frac{39}{8}}\}^{\frac{1}{3}}=(-1)^{\frac{13}{8}};$$

$$+(-\sqrt{-1})^{\frac{7}{12}}=\{(-1)^{\frac{3}{2}}\}^{\frac{7}{12}}=(-1)^{\frac{7}{8}};$$

$$+(+\sqrt{-1})^{\frac{5}{6}}=(-1)^{\frac{5}{12}}=\{(-1)^{\frac{5+16}{4}}\}^{\frac{1}{3}}=(-1)^{\frac{7}{4}}=-(-1)^{\frac{3}{4}};$$

$$+(-\sqrt{-1})^{\frac{5}{6}}=(-1)^{\frac{5}{4}}=-(-1)^{\frac{1}{4}}.$$

Further, the meaning assigned to the symbol,

$$(\cos x+\sqrt{-1}\sin x)^{\frac{m}{n}},$$

should be consistent with the above conventions and with itself, and should therefore have the same meaning as the symbol,

$$\{\cos(2r\pi+x)+\sqrt{-1}\sin(2r\pi+x)\}^{\frac{m}{n}},$$

and also the same meaning as the binomial expansion of

$$(\cos x+\sqrt{-1}\sin x)^{\frac{m}{n}},$$

when that expansion is convergent.

Now the expansion of

$$(1+\sqrt{-1}\tan x)^{\frac{m}{n}}+(1-\sqrt{-1}\tan x)^{\frac{m}{n}}$$

is convergent whenever $\tan^2 x<1$, and divergent whenever $\tan^2 x>1$.

When $\tan^2 x=1$, the series is convergent, *i.e.* the expansion of

$$(1+\sqrt{-1})^{\frac{m}{n}}+(1-\sqrt{-1})^{\frac{m}{n}}$$

is convergent when $\dfrac{m}{n}$ is positive. For the factor

$$-\frac{\left(\frac{m}{n}-r+1\right)\left(\frac{m}{n}-r\right)}{r(r+1)}$$

is $-$, if r be taken large enough, so that ultimately the terms alternate in sign, and

$$\Bigg|^{r=\infty}\left\{\frac{\frac{m}{n}\left(\frac{m}{n}-1\right)\ldots\left(\frac{m}{n}-r+1\right)}{1.2\ldots r}\right\}^2<\left(\frac{s(s-1)\ldots2.1.1.2\ldots(t-1)t}{1.2\ldots r}\right)^2,$$

where s is the integer next $> \frac{m}{n}$, $\frac{m}{n}$ being $+$, and t the integer next greater than $\left(r - \frac{m}{n} - 1\right)$, so that $s + t = r$. Now

$$\frac{s(s-1)\ldots 2.1}{(t+1)(t+2)\ldots r}$$

can be made less than any assignable quantity by taking r large enough. Hence the convergency of the series.

First, suppose x range from $-\frac{\pi}{4}$ to $+\frac{\pi}{4}$. Then our conventions require that

$$(\cos x + \sqrt{-1}\sin x)^{\frac{m}{n}} = \cos^{\frac{m}{n}} x\,(1 + \sqrt{-1}\tan x)^{\frac{m}{n}},$$

$$(\cos x - \sqrt{-1}\sin x)^{\frac{m}{n}} = \cos^{\frac{m}{n}} x\,(1 - \sqrt{-1}\tan x)^{\frac{m}{n}}.$$

For they must both be equal to unity, when $x = 0$, and both be identical with the equivalent binomial expansion, until x obtains such value as makes that expansion divergent.

Secondly, suppose x range from $+\frac{\pi}{4}$ to $+\frac{3\pi}{4}$.

Then our conventions require that

$$(\cos x + \sqrt{-1}\sin x)^{\frac{m}{n}}$$

$$= \left(\cos\frac{\pi}{2} + \sqrt{-1}\sin\frac{\pi}{2}\right)^{\frac{m}{n}} \left\{\cos\left(x - \frac{\pi}{2}\right) + \sqrt{-1}\sin\left(x - \frac{\pi}{2}\right)\right\}^{\frac{m}{n}}$$

$$= (-1)^{\frac{m}{2n}} \left\{\cos\left(x - \frac{\pi}{2}\right) + \sqrt{-1}\sin\left(x - \frac{\pi}{2}\right)\right\}^{\frac{m}{n}}.$$

Thirdly, suppose x range from $+\frac{3\pi}{4}$ to $+\frac{5\pi}{4}$. Then

$$(\cos x + \sqrt{-1}\sin x)^{\frac{m}{n}} = (-1)^{\frac{m}{n}} \{\cos(x - \pi) + \sqrt{-1}\sin(x - \pi)\}^{\frac{m}{n}}.$$

Again, suppose x range from $-\frac{3\pi}{4}$ to $-\frac{\pi}{4}$. Then

$$(\cos x + \sqrt{-1}\sin x)^{\frac{m}{n}} = \left(\cos\frac{-\pi}{2} + \sqrt{-1}\sin\frac{-\pi}{2}\right)^{\frac{m}{n}}$$

$$\left\{\cos\left(x+\frac{\pi}{2}\right) + \sqrt{-1}\sin\left(x+\frac{\pi}{2}\right)\right\}^{\frac{m}{n}}$$

$$= (-1)^{\frac{3m}{2n}}\left\{\cos\left(x+\frac{\pi}{2}\right) + \sqrt{-1}\sin\left(x+\frac{\pi}{2}\right)\right\}^{\frac{m}{n}},$$

and so in like manner in other cases.

There is true discontinuity whenever x passes through an odd multiple of $\frac{\pi}{4}$. This corelates with the circumstance that the expansions of $(1+\sqrt{-1})^{\frac{m}{n}}$ and $(\sqrt{-1}+1)^{\frac{m}{n}}$ by the Binomial Theorem have different values. Thus

$$(1+\sqrt{-1})^{\frac{1}{3}} + (1-\sqrt{-1})^{\frac{1}{3}}$$

$$= 2\left\{1 - \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{1\,.\,2} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{1}{3}-3\right)}{1\,.\,2\,.\,3\,.\,4} - \ldots\right\}$$

and, being $+$, $= 2^{\frac{7}{6}}\cos\frac{\pi}{12}$;

whereas

$$(\sqrt{-1}+1)^{\frac{1}{3}} + (-\sqrt{-1}+1)^{\frac{1}{3}} = -\sqrt{-1}(1-\sqrt{-1})^{\frac{1}{3}} + \sqrt{-1}(1+\sqrt{-1})^{\frac{1}{3}}$$

$$= 2\left\{-\frac{1}{3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{1\,.\,2\,.\,3} - \ldots\right\}$$

which, lying between 0 and $-\frac{2}{3}$, $= -2^{\frac{7}{6}}\cos\frac{5\pi}{12}$.

To recur to our Example. We have, n being odd,

$$\left(\cos\frac{\pi}{2}+\sqrt{-1}\sin\frac{\pi}{2}\right)^{\frac{1}{n}}+\left(\cos\frac{\pi}{2}-\sqrt{-1}\sin\frac{\pi}{2}\right)^{\frac{1}{n}}=(-1)^{\frac{1}{2n}}+(-1)^{\frac{3}{2n}}=0,$$

$$(\cos\pi+\sqrt{-1}\sin\pi)^{\frac{1}{n}}+(\cos\pi-\sqrt{-1}\sin\pi)^{\frac{1}{n}}=-2.$$

So that

from $\quad x=-\frac{\pi}{4}$ to $x=+\frac{\pi}{4}$, $y=2\cos\frac{x}{n}$;

from $\quad x=\frac{\pi}{4}$ to $x=\frac{3\pi}{4}$, $y=2\cos\frac{(n-1)\pi+2x}{2n}$,

or $2\cos\frac{(n+1)\pi-2x}{2n}$, according as $\frac{n-1}{2}$ is even, or $\frac{n+1}{2}$ even;

from $\quad x=\frac{3\pi}{4}$ to $x=\frac{5\pi}{4}$, $y=-2\cos\frac{\pi-x}{n}$.

From this it appears that $x=0$ makes y a maximum, and $x=\pi$ makes y a minimum, and the first and last points of the branch cutting the axis of x are respectively a minimum and maximum when $\frac{n+1}{2}$ is even, but not when $\frac{n-1}{2}$ is even. This is readily seen on tracing the curve, which we do when $n=3$, and $n=5$ (figs. 17, 18).

$$OA=AB=BC=CD=\frac{\pi}{4};$$

$$AE=2\cos\frac{\pi}{12};\ AF=2\cos\frac{5\pi}{12}=CH;$$

$$OG=2;$$

$$AE'=2\cos\frac{\pi}{20};\ AF'=2\cos\frac{9\pi}{20}=CH'.$$

110. Ex. It is required to determine the maxima and minima values of those functions of x which, when substituted for y, satisfy the equation $\qquad y^3-3ay^2=x^3.$

The curve is traced, fig. 19,

$$OA = a,\quad OB = 3a;\quad OC = a,\quad OE = 2a,$$

the ordinate of $D = \frac{a}{3}$.

In solving the equation, we find

$$(y-a)^3 - 3a^2(y-a) - 2a^3 - x^3 = 0,$$

whence the three roots, viz.:

$$y_1 = a + \left(\frac{p+\sqrt{q}}{2}\right)^{\frac{1}{3}} + \left(\frac{p-\sqrt{q}}{2}\right)^{\frac{1}{3}},$$

$$y_2 = a + \frac{-1+\sqrt{-3}}{2}\left(\frac{p+\sqrt{q}}{2}\right)^{\frac{1}{3}} + \frac{-1-\sqrt{-3}}{2}\left(\frac{p-\sqrt{q}}{2}\right)^{\frac{1}{3}},$$

$$y_3 = a + \frac{-1-\sqrt{-3}}{2}\left(\frac{p+\sqrt{q}}{2}\right)^{\frac{1}{3}} + \frac{-1+\sqrt{-3}}{2}\left(\frac{p-\sqrt{q}}{2}\right)^{\frac{1}{3}};$$

where $$p = 2a^3 + x^3,$$

$$q = (2a^3 + x^3)^2 - 4a^6 = 4a^3x^3 + x^6.$$

The function y_1 is real always, and consequently applies to the branches QR' and BP. So it must have true discontinuity somewhere. Art. 109 proves that this may take place at the points whereat the series of which y_1 is the envelop becomes divergent. This is confirmed by the fact that, when $2a^3 + x^3 = 0$, $y_1 = a$, a point in the branch OR.

So that when x is between the real roots of the equation,

$$(2a^3 + x^3)^2 = 4a^6 - (2a^3 + x^3)^2,$$

i.e., when x^3 lies between $-(2+\sqrt{2})\,a^3$ and $-(2-\sqrt{2})\,a^3$, y_1 is the ordinate of the branch OR, and is a maximum at one end, and a minimum at the other. For other values of x, y_1 is the ordinate of the branches QS', SP, and has no finite value which is a maximum or a minimum.

Again, when $\sqrt{-q}$ is small, we have, expanding according to ascending powers of $\sqrt{-q}$,

$$y_2 = a - \left(\frac{p}{2}\right)^{\frac{1}{3}} + \frac{\sqrt{-3}}{3}\left(\frac{p}{2}\right)^{-\frac{2}{3}}\left(\frac{\sqrt{q}}{2}\right)^{\frac{1}{3}} + \dots$$

$$= a - \left(\frac{p}{2}\right)^{\frac{1}{3}} + (-1)^{\frac{1}{2}}(-1)^{\frac{1}{6}}\frac{\sqrt{3}}{3}\left(\frac{p}{2}\right)^{-\frac{2}{3}}\left(\frac{\sqrt{-q}}{2}\right)^{\frac{1}{3}} + \dots$$

$$= a - \left(\frac{p}{2}\right)^{\frac{1}{3}} + (-1)^{\frac{1}{2}}\{(-1)^{\frac{9}{2}}\}^{\frac{1}{3}}\frac{\sqrt{3}}{3}\left(\frac{p}{2}\right)^{-\frac{2}{3}}\left(\frac{\sqrt{-q}}{2}\right)^{\frac{1}{3}} + \dots$$

$$= a - \left(\frac{p}{2}\right)^{\frac{1}{3}} + \dots$$

So that, near R, y_2 is the ordinate of the branch RS, and near O is the ordinate of the branch OD.

When $p = 0$, we have

$$y_2 = a + \frac{-1+\sqrt{-3}}{2}(-1)^{\frac{1}{6}}\left(\frac{\sqrt{-q}}{2}\right)^{\frac{1}{3}} + \frac{-1-\sqrt{-3}}{2}(-1)^{\frac{5}{6}}\left(\frac{\sqrt{-q}}{2}\right)^{\frac{1}{3}}$$

$$= a - (-1)^{\frac{1}{2}}(-1)^{\frac{1}{2}}\sqrt{3}\left(\frac{\sqrt{-q}}{2}\right)^{\frac{1}{3}} = a + \sqrt{3}\left(\frac{\sqrt{-q}}{2}\right)^{\frac{1}{3}}.$$

We conclude that y_2 is the ordinate of RS and points on OD near O, and has a minimum value for the points R, O, and a maximum for the point S. In like manner y_3 is the ordinate of OS' and points on RO near R, and has a maximum value for the points R, O, and a minimum value for the point S'.

111. Ex. It is required to determine the maxima and minima values of the z ordinate of the outside sheet above the plane of xy of the surface,

$$(x^2 + y^2 + z^2 - d^2)^2 = 4a^2(x^2 + y^2).$$

For that portion of the surface, and for none other,

$$z = \sqrt{d^2 - x^2 - y^2 + 2a\sqrt{x^2 + y^2}},$$

$$2z\frac{dz}{dx} = -2x + \frac{2ax}{\sqrt{x^2+y^2}},$$

$$2z\frac{dz}{dy} = -2y + \frac{2ay}{\sqrt{x^2+y^2}}.$$

And when $x = 0$, $y = 0$, $z = d$, we have

$$\frac{k\frac{dz}{dx} + \frac{dz}{dy}}{\sqrt{1+k^2}} = \frac{akx + ay}{z\sqrt{1+k^2}\sqrt{x^2+y^2}},$$

which always changes sign, being then truly discontinuous, as x, y, pass through zero for their particular simultaneous values. Therefore $z = d$ is a minimum. It can be shewn that $z = 0$, given by

$$x^2 + y^2 = (a + \sqrt{a^2+d^2})^2,$$

is another minimum, and $z = \sqrt{a^2+d^2}$ a maximum.

112. Suppose that, for all values of x between x_0 and x_1, y and its derived functions are such that in the equation,

$$f\left\{x,\ y+\delta y,\ \frac{d(y+\delta y)}{dx},\ \dots\ \frac{d^n(y+\delta y)}{dx^n}\right\} - f\left(x, y, \frac{dy}{dx}, \dots\ \frac{d^n y}{dx^n}\right) = \delta f,$$

(wherein δy is the unknown quantity, and δf is any function of x,) $f\left(x, y, \frac{dy}{dx}, \dots\ \frac{d^n y}{dx^n}\right)$ has real value. This is a differential equation, satisfied by $\delta y = 0$, $\frac{d\delta y}{dx} = 0$, ... $\frac{d^n \delta y}{dx^n} = 0$, and which will in general have many systems of real solutions. Of these solutions there will in general be at least one in which δy, $\frac{d\delta y}{dx}$, ... continuously change from zero as δf continuously changes from zero, whatever be the form of δf.

In such a solution δy, $\frac{d\delta y}{dx}$, ... may be, as x changes, truly discontinuous, but must be such that, for a given value of x, they are not truly discontinuous functions of the other variables involved, such as those which enable δf to vary in form.

If, when y, $\frac{dy}{dx}$, ... $\frac{d^n y}{dx^n}$, are particular functions of x, whether truly discontinuous or not, the above equation has, when δf is a function never having a negative value, no such real solution, then y, $\frac{dy}{dx}$, ... $\frac{d^n y}{dx^n}$, are said to be those (truly discontinuous if necessary) functions which render $f(x, y, \frac{dy}{dx}, ...)$ a maximum. It is obvious that their value must be sought by the methods of Arts. 105 to 107. Such a maximum we shall call a broken maximum.

If when y is a particular not truly discontinuous function of x, the above equation has, when δf is a function never having a negative value, no such real solution such that δy is a not truly discontinuous function of x, then y is said to be that not truly discontinuous function which renders $f(x, y, \frac{dy}{dx}, ...)$ a maximum. Such a maximum we shall call an unbroken maximum.

If in the preceding, 'positive' be substituted for 'negative', and 'minimum' for 'maximum', we obtain definitions for 'a broken minimum' and 'an unbroken minimum.' Functions, when they are both maximum and minimum, may then be functions either constancy or isolate in value.

113. Observe that, if neither y nor δy are truly discontinuous functions of x, $(y + \delta y)$ is a not truly discontinuous function of x; and that if only one of the two functions, y, δy, is a truly discontinuous function of x, then $(y + \delta y)$ is a truly discontinuous function of x; and that if both y, δy are truly discontinuous functions of x, then $(y + \delta y)$ may or may not be a truly discontinuous function of x.

114. It may happen that $\frac{dy}{dx}$, $\frac{d^2 y}{dx^2}$, ... $\frac{d^n y}{dx^n}$, are some of them truly discontinuous functions, although y is not so. In this case y will always have false discontinuity, provided the truly discontinuous derived function is finite. So in like manner, $\frac{d\delta y}{dx}$,

$\frac{d^2\delta y}{dx^2}, \dots \frac{d^n\delta y}{dx^n}$, may some of them, when finite, be truly discontinuous, although δy is not so. In this case δy will have false discontinuity. But always, both in truly discontinuous and in other solutions, $\delta y, \frac{d\delta y}{dx}, \dots \frac{d^n\delta y}{dx^n}$, must each one of them continuously change from zero, as δf continuously changes from zero, in the solution of the equation of Art. 112, there contemplated.

115. When the condition of Art. 112, viz. that the equation of that article has always one real solution which continuously changes from zero as δf continuously changes from zero, is not satisfied; then, if in the equation,

$$f\left\{x,\ y+\delta y,\ \frac{d(y+\delta y)}{dx}, \dots \frac{d^n(y+\delta y)}{dx^n}\right\} - f\left(x, y, \frac{dy}{dx}, \dots \frac{d^ny}{dx^n}\right) = \delta f,$$

$\delta y, \frac{d\delta y}{dx}, \dots \frac{d^n\delta y}{dx^n}$, be all taken any functions whatever continuously changing from zero, δf will be zero when they are all zero, and must then be truly discontinuous both ways, *i.e.* be isolate in value. In this case the form of f is truly discontinuous for those values of y and its derived functions. Such forms should be always separately examined to see what their nature is.

116. If we express the initial value of

$$\frac{f\left(x,\ y+\delta y,\ \frac{dy}{dx}, \dots \frac{d^ny}{dx^n}\right) - f\left(x, y, \frac{dy}{dx}, \dots \frac{d^ny}{dx^n}\right)}{\delta y}$$

$$\text{by } N \text{ or } N\left(x, y, \frac{dy}{dx}, \dots \frac{d^ny}{dx^n}\right),$$

$$\text{of } \frac{f\left\{x,\ y,\ \frac{d(y+\delta y)}{dx}, \frac{d^2y}{dx^2}, \dots \frac{d^ny}{dx^n}\right\} - f\left(x, y, \frac{dy}{dx}, \dots \frac{d^ny}{dx^n}\right)}{\frac{d\delta y}{dx}}$$

$$\text{by } P_1 \text{ or } P_1\left(x, y, \frac{dy}{dx}, \dots \frac{d^ny}{dx^n}\right),$$

……………………………………………………………

and of

$$\frac{f\left\{x,\ y,\ \frac{dy}{dx},\ \dots\ \frac{d^{n-1}y}{dx^{n-1}},\ \frac{d^n(y+\delta y)}{dx^n}\right\}-f\left(x,\ y,\ \frac{dy}{dx},\ \dots\ \frac{d^ny}{dx^n}\right)}{\frac{d^n\delta y}{dx^n}}$$

$$\text{by } P_n \text{ or } P_n\left(x,\ y,\ \frac{dy}{dx},\ \dots\ \frac{d^ny}{dx^n}\right),$$

we tacitly assume that these initial values are independent of the particular form, as functions of x, which δy, $\frac{d\delta y}{dx}$, ... have, and so have the same value as if these forms were independent of x. The assumption is readily justified by recognising that, if, for a particular form of δy, any such initial value were not the same as when δy has the constant value which is the minimum value of the particular form, by taking δy smaller a different initial value would be obtained. So the mode of expression is justifiable.

117. We have, employing the functions defined in Art. 116, the equation

$$\begin{aligned}\delta f = {} & N\left(x,\ y,\ \frac{dy}{dx},\ \dots\ \frac{d^ny}{dx^n}\right)\delta y \\ & + P_1\left(x,\ y+\delta y,\ \frac{dy}{dx},\ \dots\ \frac{d^ny}{dx^n}\right)\frac{d\delta y}{dx} \\ & + P_2\left\{x,\ y+\delta y,\ \frac{d(y+\delta y)}{dx},\ \dots\ \frac{d^ny}{dx^n}\right\}\frac{d^2\delta y}{dx^2} \\ & + \dots + P_n\left\{x,\ y+\delta y,\ \frac{d(y+\delta y)}{dx},\ \dots\ \frac{d^{n-1}(y+\delta y)}{dx^{n-1}},\ \frac{d^ny}{dx^n}\right\}\frac{d^n\delta y}{dx^n}\end{aligned}$$

holding initially. So that, if none of the functions P_1, P_2, ... P_n, are truly discontinuous, initially

$$\delta f = N\delta y + P_1\frac{d\delta y}{dx} + \dots + P_n\frac{d^n\delta y}{dx^n},$$

and, in like manner,

$$\delta\left[f\left\{x, y-\delta y, \frac{d(y-\delta y)}{dx}, \ldots \frac{d^n(y-\delta y)}{dx^n}\right\}\right]$$

$$= N\left\{x, y-\delta y, \frac{d(y-\delta y)}{dx}, \ldots \frac{d^n(y-\delta y)}{dx^n}\right\}\delta y$$

$$+ P_1\left\{x, y, \frac{d(y-\delta y)}{dx}, \ldots \frac{d^n(y-\delta y)}{dx^n}\right\}\frac{d\delta y}{dx}$$

$$+ \ldots + P_n\left\{x, y, \frac{dy}{dx}, \ldots \frac{d^{n-1}y}{dx^{n-1}}, \frac{d^n(y-dy)}{dx^n}\right\}\frac{d^n\delta y}{dx^n},$$

and similarly for $\delta\left[f\left\{x, y+\delta y, \frac{d(y+\delta y)}{dx}, \ldots\ldots \frac{d^n(y+\delta y)}{dx^n}\right\}\right]$.

Upon the signs of these expressions will depend the distinction between maxima and minima.

118. Supposing f to contain only x, y, $\frac{dy}{dx}$, we obtain from the equation,

$$N\delta y + P_1\frac{d\delta y}{dx} = \delta f,$$

$$\delta y = e^{-\int\frac{N}{P_1}dx}\int_\xi^x \frac{e^{\int\frac{N}{P_1}dx}}{P_1}\delta f dx,$$

$$\frac{d\delta y}{dx} = -\frac{N}{P_1}\delta y + \frac{\delta f}{P_1}.$$

Now if $\int\frac{N}{P_1}dx$ never become infinite, δf can be so chosen as to have always the same sign, and that $\frac{e^{\int\frac{N}{P_1}dx}}{P_1}\delta f$ be never infinite even when P_1 vanishes. In this case a not truly discontinuous value will always be the value given to δy in the above equation.

But if $\int \frac{N}{P_1} dx$ become infinite, the limits should be so chosen as not to include between them values of x for which this occurs. We may do this when x_0', x_1' are successive values of x for which this occurs, by expressing δy thus,

$$\delta y = e^{-\int_{\xi_1}^{x} \frac{N}{P_1} dx} \int_{\xi}^{x} \frac{e^{\int_{\xi_1}^{x} \frac{N}{P_1} dx}}{P_1} \delta f dx,$$

where x ranges from x_0' to x_1', and ξ_1 is intermediate to these values.

This equation shows that if both $e^{-\int_{\xi_1}^{x_0'} \frac{N}{P_1} dx}$ and $e^{-\int_{\xi_1}^{x_1'} \frac{N}{P_1} dx}$ are infinite, which will not be the case unless $\Big|^{x_0'} \frac{N}{P_1}$, $\Big|^{x_1'} \frac{N}{P_1}$ have opposite signs, it may be impossible so to select ξ, ξ_1, δf that δy be finite both when $x = x_0'$, and $x = x_1'$ (δf never changing sign), otherwise than by supposing that $\delta f = 0$ throughout, and $\frac{\delta f}{P_1}$ is finite at the points corresponding to x_0', x_1'.

In such a case we may expect to find an unbroken maximum or minimum. The nature will be determined by investigating what the effect of an indefinitely small change, at the points corresponding to x_0', x_1', is.

Again, if two portions of a curve form unbroken maxima or minima, and if in the portion which joins them P_1 never change sign, and $\int_{\xi_1}^{x} \frac{N}{P_1} dx$ never become infinite, δy for that portion cannot vanish twice without vanishing altogether, and we shall have in the falsely discontinuous curve formed by the three portions an unbroken maximum or minimum. A particular instance of unbroken maximum or minimum may obviously be expected to occur whenever $\int_{x_0}^{x_1} f dx$ is a maximum or a minimum.

It is manifest that if a portion of a curve which is an unbroken maximum joined on to a portion of a curve which is an unbroken minimum, the combined curve would be both an unbroken maximum and an unbroken minimum. It would be not appropriate to say that such a curve gave an isolate or a conjugate value to the function f. For in all cases, except when the broken maximum or minimum happens to be unbroken, by allowing f to have change of sign, the equation, $N\delta y + P_1 \frac{d\delta y}{dx} = \delta f$, can be so satisfied so that δy have any not truly discontinuous value we please.

119. Generally we have to consider whether the equation

$$N\delta y + P_1 \frac{d\delta y}{dx} + P_2 \frac{d^2\delta y}{dx^2} + \dots + P_n \frac{d^n\delta y}{dx^n} = \delta f$$

can be satisfied, when δf is invariable in sign, so that δy is not truly discontinuous. If this cannot be done in any particular case, that case may be expected to present an instance of a maximum or a minimum. As before stated, a broken maximum or minimum may often be expected to be given by the system

$$N = P_1 = P_2 = \dots = P_n = 0.$$

If y only be required to be not truly discontinuous, we may in many cases put

$$P_2 = P_3 = \dots = P_n = 0,$$

$$N\delta y + P_1 \frac{d\delta y}{dx} = \delta f,$$

and proceed as in Art. 118. But if the solution of the equations $P_2 = P_3 = \dots = P_n$ require y to be truly discontinuous, the method will not succeed.

If it be required that a particular selection from

$$\frac{d\delta y}{dx}, \frac{d^2\delta y}{dx^2}, \dots \frac{d^n dy}{dx^n}$$

should be not truly discontinuous as well as δy, we might proceed by equating to zero as many of the expressions N, P_1, P_2, ... P_n, as the circumstances of the case permit, leaving a linear differential equation, upon the possibility of whose solution would the criterion of a maximum or a minimum depend. A problem so to be solved might be conveniently expressed as follows.

It is required to find the values of $\left(\int\right)^r y\,dx^r, \ldots y, \ldots \frac{d^s y}{dx^s}$, when

$$f\left\{x,\ \left(\int\right)^r y\,dx^r,\ \left(\int\right)^{r-1} y\,dx^{r-1}, \ldots\ \left(\int\right) y\,dx,\ y,\ \frac{dy}{dx}, \ldots \frac{d^s y}{dx^s}\right\},$$

has a maximum or a minimum value, wherein

$$y = \int_\xi^x \frac{dy}{dx};\quad \left(\int\right)^m y\,dx^m = \int_\xi^x \left(\int\right)^{m-1} y\,dx^{m-1}\,dx; \ldots$$

$$\frac{d^n y}{dx^n} = \int_\xi^x \frac{d^{n+1} y}{dx^{n+1}}\,dx; \ldots$$

ξ being a continuous constant: it being understood that, for all other differential and integral coefficients of y, ξ may be a truly discontinuous constant.

120. Ex. The function

$$y^2 + \left(\frac{dy}{dx} - b\right)^2$$

has a broken minimum value when

$$\begin{aligned} y = \Big|^{a=0} &\left\{bx - \frac{ab\sqrt{(x-a)^2}}{2(x-a)} - \frac{ab\sqrt{(x-2a)^2}}{2(x-2a)} - \ldots - \frac{ab\sqrt{\left(x-\frac{c}{a}\right)^2}}{2\left(x-\frac{c}{a}\right)}\right. \\ &\left. - \frac{ab\sqrt{(x+a)^2}}{2(x+a)} - \frac{ab\sqrt{(x+2a)^2}}{2(x+2a)} - \ldots - \frac{ab\sqrt{\left(x+\frac{c}{a}\right)^2}}{2\left(x+\frac{c}{a}\right)}\right], \end{aligned}$$

$$\frac{dy}{dx} = b,$$

where c is a finite constant so chosen as to be a multiple of a^2. (Vide Art. 87.) Figure 20 represents, for a small value of a, the curve whose equation is above given.

To determine whether a particular value of y renders

$$y^2 + \left(\frac{dy}{dx} - b\right)^2$$

an unbroken minimum, we must investigate whether the differential equation

$$y\delta y + \left(\frac{dy}{dx} - b\right)\frac{d\delta y}{dx} = \delta f,$$

where δf is never +, admits of a solution, wherein y is not truly discontinuous, for any such form for δf.

If we take

$$y = \frac{b\,(e^x - e^{-x})}{e^h + e^{-h}} + k; \quad \frac{dy}{dx} = \frac{b\,(e^x + e^{-x})}{e^h + e^{-h}};$$

x being supposed to range from $-h$ to $+h$, we have

$$\delta y = e^{-\int \frac{N}{P} dx} \int_\xi^x \frac{e^{\int \frac{N}{P} dx}}{P}\, \delta f dx,$$

where $$\frac{N}{P} = \frac{b\,(e^x - e^{-x}) + k\,(e^h + e^{-h})}{b\,(e^x + e^{-x}) - b\,(e^h + e^{-h})}.$$

P has always the same sign, and is $-$: so that

$$\frac{e^{\int \frac{N}{P} dx}}{P} = -e^{-\log -P + \int \frac{N}{P} dx} = -e^{\int \frac{1}{P}\left(N - \frac{dP}{dx}\right) dx}.$$

Now $N - \dfrac{dP}{dx} = k$, so that

$$\delta y = -e^{-\int_{\xi_1}^x \frac{N}{P} dx} \int_\xi^x e^{\int_{\xi_1}^x \frac{k}{P} dx}\, \delta f dx,$$

as P vanishes when $x = +h$, and when $x = -h$, if N never change sign, either $e^{-\int_{\xi_1}^h \frac{N}{P} dx}$ or $e^{-\int_{\xi_1}^{-h} \frac{N}{P} dx}$ vanishes, and by taking $\delta f = -\sqrt{N^2 \delta c^2}$, we obtain $\delta y = \pm\, \delta c$, $\dfrac{d\delta y}{dx} = 0$, the sign being chosen so that y and δy have opposite signs.

But if N change sign, it must change from $-$ to $+$, so that $-\int_{\xi_1}^{h} \frac{N}{P} dx$ is $+$, and also $\int_{-h}^{\xi_1}$, and, as the expression under $\int_{\xi}^{x}$ does not change sign, δy must be truly discontinuous, unless δf always zero.

We infer that if the curve AB (fig. 21) lie entirely on one side of the axis, $y^2 + \left(\frac{dy}{dx} - b\right)^2$ is not a minimum, for it is throughout less in the parallel curve $A'B'$ nearer the axis of x.

But if it cross the axis, as at C (fig. 22), it is in the position of an unbroken minimum, for an indefinitely small change in $\frac{dy}{dx}$ at A or B increases $y^2 + \left(\frac{dy}{dx} - b\right)^2$.

If we investigate whether $y = b \sin x$ renders

$$y^2 + \left(\frac{dy}{dx} - b\right)^2$$

an unbroken minimum, we may confine our attention to the range from 0 to 2π for x. Now the equation

$$\delta y = e^{\int \frac{\sin x dx}{1 - \cos x}} \int_{\xi}^{x} \frac{e^{-\int \frac{\sin x dx}{1 - \cos x}}}{-(1 - \cos x)} \delta f dx$$

shows that by assuming, δc being $+$,

$$\delta y = + (1 - \cos x)(x - \xi)\, \delta c,$$

$$\frac{d\delta y}{dx} = + \sin x\, (x - \xi)\, \delta c + (1 - \cos x)\, \delta c,$$

we obtain a curve, unbroken, throughout which

$$y^2 + \left(\frac{dy}{dx} - b\right)^2$$

is less than when $y = p \sin x$, except when $x = 0$, or $x = 2\pi$, when it has the same value. We trace the second curve in a few cases (figs. 23, 24, 25).

$y=0$ will be found to give an unbroken minimum when the range is infinite, and also when the range is finite, provided $\frac{dy}{dx}=b$ at the extremities, A, B (fig. 26).

Suppose (fig. 27) from A to B,

$$y=\frac{b\,(e^{x+2a}-e^{-x})}{e^{a+h}+e^{a-h}}+k;\quad \frac{dy}{dx}=\frac{b\,(e^{x+2a}+e^{-x})}{e^{a+h}+e^{a-h}};$$

and from B to C, $\frac{dy}{dx}$ is constant, and from C to D,

$$y=\frac{b\,(e^{x-2a'}-e^{-x})}{e^{-a'+h}+e^{-a'-h}};\quad \frac{dy}{dx}=\frac{b\,(e^{x-2a'}+e^{-x})}{e^{-a'+h}+e^{-a'-h}};$$

AB, and CD, are unbroken minima, and BC contains no point whereat $\left(\frac{dy}{dx}-b\right)$ vanishes. So $ABCD$ is a position of unbroken minimum. It is evident from this example that falsely discontinuous curves without end, and defying classification, can be assigned in any particular case, all fulfilling the tests of unbroken minima or maxima, whenever simple curves fulfilling the same tests can be found.

Also it is evident that, where broken maxima and minima exist, *i.e.*, wherever the system,

$$N=P_1=P_2=\dots\dots=0,$$

points to a maximum or minimum, and the maximum or minimum pointed to is not continuously satisfiable by y, $\frac{dy}{dx}$, ... the existence respectively of unbroken maxima and minima may be inferred.

121. Suppose it is required to determine the form of the function y, so that

$$u=\int_{x_0}^{x_1}f\left(x,\, y,\, \frac{dy}{dx},\, \dots\, \frac{d^n y}{dx^n}\right)dx$$

be a maximum or a minimum. If y, or any of its differential coefficients, as $\frac{d^r y}{dx^r}$, may be any truly discontinuous function of x, the lower limit in the equations,

$$y = \int_\xi^x \frac{dy}{dx} dx, \quad \frac{d^r y}{dx^r} = \int_{\xi'}^x \frac{d^{r+1} y}{dx^{r+1}} dx, \ldots$$

may be a discontinuous constant. So y, $\frac{d^r y}{dx^r}$, and as many differential coefficients as may be truly discontinuous, may be regarded as perfectly independent of the rest. Such problems merge into the problem of finding the maximum and minimum value of

$$\int_{x_0}^{x_1} f\left(x, z_1, \frac{dz_1}{dx}, \ldots z_2, \frac{dz_2}{dx}, \ldots z_3, \frac{dz_3}{dx}, \ldots\right) dx,$$

where z_1, z_2, z_3 are independent functions. When not otherwise stated, it will always be supposed that ξ, ξ' are continuous constants in the above equations, so that y, $\frac{dy}{dx}$,... cannot be truly discontinuous through ξ, ξ' being so, although they may be infinite.

122. As in Art. 117, if any of the functions

$$N, P_1, \ldots P_n,$$

are truly discontinuous functions of y, $\frac{dy}{dx}$, ... $\frac{d^n x}{dx^n}$,

$$\int_{x_0}^{x_1} \delta f dx$$

may change in value abruptly, in consequence of a change in δy, in a manner analogous to what takes place in our examples of Arts. 108—111. So if in any case any of the functions, N, P_1, P_2, ... P_n, evince true discontinuity, that case should be separately examined in order to determine what happens at the true discontinuity.

123. We have, barring cases of true discontinuity,

$$\delta u = \int_{x_0}^{x_1} \delta f dx = \int_{x_0}^{x_1} f\left\{x,\ y + \frac{\delta y}{dx},\ \frac{d(y+\delta y)}{dx},\ \dots\ \frac{d^n(y+\delta y)}{dx^n}\right\} dx$$

$$- \int_{x_0}^{x_1} f\left(x,\ y,\ \frac{dy}{dx}, \dots\ \frac{d^n y}{dx^n}\right) dx$$

$$= \int_{x_0}^{x_1} \left(N\delta y + P_1 \frac{d\delta y}{dx} + P_2 \frac{d^2\delta y}{dx^2} + \dots + P_n \frac{d^n \delta y}{dx^n}\right) dx$$

initially, for all such forms of δy, $\frac{d\delta y}{dx}$, ... $\frac{d^n\delta y}{dx^n}$ as are initially evanescent. So that

$$\delta u = \int_{x_0}^{x_1} \frac{d^n\delta y}{dx^n} \left\{P_n - \frac{d^{-1}P_{n-1}}{dx^{-1}} + \frac{d^{-2}P_{n-2}}{dx^{-2}} - \dots + (-1)^n \frac{d^{-n}N}{dx^{-n}}\right\} dx$$

$$+ \Big|_{x_0}^{x_1} \left[\frac{d^{n-1}\delta y}{dx^{n-1}} \left\{\frac{d^{-1}P_{n-1}}{dx^{-1}} - \frac{d^{-2}P_{n-2}}{dx^{-2}} + \dots + (-1)^{n-1} \frac{d^{-n}N}{dx^{-n}}\right\}\right.$$

$$+ \frac{d^{n-2}\delta y}{dx^{n-2}} \left\{\frac{d^{-1}P_{n-2}}{dx^{-1}} - \frac{d^{-2}P_{n-3}}{dx^{-2}} + \dots + (-1)^{n-2} \frac{d^{-n+1}N}{dx^{-n+1}}\right\}$$

$$+ \dots + \frac{d^2\delta y}{dx^2} \left(\frac{d^{-1}P_2}{dx^{-1}} - \frac{d^{-2}P_1}{dx^{-2}} + \frac{d^{-3}N}{dx^{-3}}\right)$$

$$\left. + \frac{d\delta y}{dx} \left(\frac{d^{-1}P_1}{dx^{-1}} - \frac{d^{-2}N}{dx^{-2}}\right) + \delta y \frac{d^{-1}N}{dx^{-1}}\right],$$

and this must vanish initially when u is a maximum or a minimum; for a change in the sign of δy changes its sign.

Now, N, P_1, P_2, ... P_n, being all not truly discontinuous, the same will be true of any negative differential coefficient of any one of them. And if $\frac{d^r\delta y}{dx^r}$ be taken $= \Big|^{a=\infty} \delta c \sin ax$, where δc does not involve x, all negative differential coefficients of it can be made less than any assignable quantity. We must have, therefore, initially,

$$0 = \int_{x_0}^{x_1} \frac{d^n\delta y}{dx^n} \left\{P_n - \frac{d^{-1}P_{n-1}}{dx^{-1}} + \frac{d^{-2}P_{n-2}}{dx^{-2}} - \dots + (-1)^n \frac{d^{-n}N}{dx^{-n}}\right\} dx;$$

$$0 = \int_{x_0}^{x_1} \frac{d^n \delta y}{dx^n} (P_n - \ldots)\, dx + \Big|_{x_0}^{x_1} \frac{d^{n-1}\delta y}{dx^{n-1}} \left(\frac{d^{-1}P_{n-1}}{dx^{-1}} - \ldots + (-1)^{n-1} \frac{d^{-n}N}{dx^{-n}} \right);$$

$$0 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

So that

$$0 = \Big|_{x_0}^{x_1} \left\{ \frac{d^{-1}P_{n-1}}{dx^{-1}} - \frac{d^{-2}P_{n-2}}{dx^{-2}} + \ldots\ldots + (-1)^{n-1} \frac{d^{-n}N}{dx^{-n}} \right\} \frac{d^{n-1}\delta y}{dx^{n-1}};$$

$$0 = \Big|_{x_0}^{x_1} \left\{ \frac{d^{-1}P_{n-2}}{dx^{-1}} - \frac{d^{-2}P_{n-3}}{dx^{-2}} + \ldots\ldots + (-1)^{n-2} \frac{d^{-n+1}N}{dx^{-n+1}} \right\} \frac{d^{n-2}\delta y}{dx^{n-2}}:$$

$$0 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$0 = \Big|_{x_0}^{x_1} \frac{d^{-1}N}{dx^{-1}} \delta y.$$

124. Representing by I, or $I(x)$, (Art. 98), any function of x, having false discontinuity, whose general value is zero, but which has other values at particular values, finite in number, of x, we shall have

$$\int_{x_0}^{x_1} I(x)\, \delta X dx = 0,$$

whatever finite value δX have, and so the equation

$$\int_{x_0}^{x_1} \frac{d^n \delta y}{dx^n} \left\{ P_n - \frac{d^{-1}P_{n-1}}{dx^{-1}} + \frac{d^{-2}P_{n-2}}{dx^{-2}} - \ldots + (-1)^n \frac{d^{-n}N}{dx^{-n}} \right\} dx = 0$$

is satisfied when

$$P_n - \frac{d^{-1}P_{n-1}}{dx^{-1}} + \frac{d^{-2}P_{n-2}}{dx^{-2}} - \ldots + (-1)^n \frac{d^{-n}N}{dx^{-n}} = I(x).$$

In differentiating this, a function of the form $I(x)$ should be introduced at each differentiation. If, on account of any geometrical interpretation, or otherwise, infinite values for δp_n be considered admissible, we must investigate whether such values can render $\int_{x_0}^{x_1} \delta p_n I dx$ other than zero, carefully remembering that the admission of such values invalidates our method (and every method which employs, either with acknowledgment or without acknowledgment, Taylor's Theorem) *ab initio*. For it is necessary, from the outset, to restrict δy, δp, ... to such values as are initially evanescent.

125. It should be observed that, since

$$\int_{x_0}^{x_1} \frac{d^n\delta y}{dx^n}\left\{P_n - \frac{d^{-1}P_{n-1}}{dx^{-1}} + \frac{d^{-2}P_{n-2}}{dx^{-2}} + \ldots + (-1)^n \frac{d^{-n}N}{dx^{-n}}\right\} dx$$

$$\equiv \Big|_{x_0}^{x_1} \frac{d^n\delta y}{dx^n}\left\{\frac{d^{-1}P_n}{dx^{-1}} - \frac{d^{-2}P_{n-1}}{dx^{-2}} + \frac{d^{-3}P_{n-2}}{dx^{-3}} + \ldots + (-1)^n \frac{d^{-n-1}N}{dx^{-n-1}}\right\}$$

$$- \int_{x_0}^{x_1} \frac{d^{n+1}\delta y}{dx^{n+1}}\left\{\frac{d^{-1}P_n}{dx^{-1}} - \frac{d^{-2}P_{n-1}}{dx^{-2}} + \ldots + (-1)^n \frac{d^{-n-1}N}{dx^{-n-1}}\right\} dx,$$

if we refuse to consider infinite values of $\frac{d^{n+1}\delta y}{dx^{n+1}}$ or $\frac{d\delta p_n}{dx}$, admissible, we shall obtain the equation

$$\frac{d^{-1}P_n}{dx^{-1}} - \frac{d^{-2}P_{n-1}}{dx^{-2}} + \ldots + (-1)^n \frac{d^{-n-1}N}{dx^{-n-1}} = I_1(x).$$

The solutions of this equation which are not also solutions of

$$P_n - \frac{d^{-1}P_{n-1}}{dx^{-1}} + \ldots + (-1)^n \frac{d^{-n}N}{dx^{-n}} = I(x),$$

will enable us to render $\int_{x_0}^{x_1} \delta p_n \left(P_n - \frac{d^{-1}P_{n-1}}{dx^{-1}} + \ldots\right) dx$ other than zero for values of δp_n for which $\frac{d\delta p_n}{dx}$ is infinite, and will be referred to in subsequent Articles.

It should be borne in mind that, though δy, δp_1, ... δp_n must be initially evanescent, no such initial necessity applies to $\frac{d\delta p_n}{dx}$.

126. In the last three Articles no restrictions limit in generality δy, δp_1, ... But if it be required to determine what function y must be of x in order that

$$u \equiv \int_{x_0}^{x_1} f\left(x,\, y,\, \frac{dy}{dx},\, \ldots\, \frac{d^n y}{dx^n}\right) dx \equiv \int_{x_0}^{x_1} f(x, y, p_1, \ldots p_n)\, dx$$

be a maximum or a minimum, when the equations

$$c_1=\int_{x_0}^{x_1}\phi_1\left(x,\ y,\ \frac{dy}{dx},\ \dots\ \frac{d^ny}{dx^n}\right)dx,$$

$$c_2=\int_{x_0}^{x_1}\phi_2\left(x,\ y,\ \frac{dy}{dx},\ \dots\ \frac{d^ny}{dx^n}\right)dx,$$

$$\dots=\dots\dots\dots\dots\dots\dots\dots\dots$$

$$c_r=\int_{x_0}^{x_1}\phi_r\left(x,\ y,\ \frac{dy}{dx},\ \dots\ \frac{d^ny}{dx^n}\right)dx,$$

c_1, c_2, ... c_r being constants, hold, we must have

$$u=\int_{x_0}^{x_1}\{f+\lambda_1\phi_1+\lambda_2\phi_2+\dots+\lambda_r\phi_r\}\,dx,$$

a maximum or a minimum, whatever arbitrary constants $\lambda_1, \lambda_2, \dots \lambda_r$ may be, for all variations consistent with the given equations. So r arbitrary constants will be introduced into the equations of Art. 123. In general the elimination of these arbitrary quantities will reduce the number of these equations by r, which are replaced by our r equations $c_1=\dots$, ..., $c_r=\dots$.

127. It is obvious that cases, wherein

$$u=\int_{x_0}^{x_1}f\left\{x,\ (\textstyle\int)^r y\,dx^r,\ (\textstyle\int)^{r-1}y\,dx^{r-1},\ \dots\ \textstyle\int y\,dx,\ y,\ \dfrac{dy}{dx},\ \dots\ \dfrac{d^ny}{dx^n}\right\}dx,$$

reduce to cases of Art. 121 by assuming

$$(\textstyle\int)^r y\,dx^r=z, \text{ which gives}$$

$$u=\int_{x_0}^{x_1}f\left\{x,\ z,\ \frac{dz}{dx},\ \dots\ \frac{d^{n+r}z}{dx^{n+r}}\right\}dz.$$

128. When $y, \dfrac{dy}{dx}, \dots \dfrac{d^ny}{dx^n}$ satisfy the equations of Art. 123, we shall have, initially,

$$\int_{x_0}^{x_1}\left[f\left(x,\ y,\ \frac{dy}{dx},\ \dots\ \frac{d^ny}{dx^n}\right)\right.$$
$$\left.-f\left\{x,\ y-\delta y,\ \frac{d(y-\delta y)}{dx},\ \dots\ \frac{d^n(y-\delta y)}{dx^n}\right\}\right]dx$$

$$= \int_{x_0}^{x_1} \left\{ N(y - \delta y)\,\delta y + P_1(y - \delta y)\,\frac{d\delta y}{dx} + \ldots + P_n(y - \delta y)\,\frac{d^n\delta y}{dx^n} \right\} dx$$

$$= \int_{x_0}^{x_1} \delta f(y - \delta y)\, dx \text{ suppose,}$$

$N(y - \delta y)$, $P_1(y - \delta y)$, ... representing that $(y - \delta y)$ is substituted for y in N, P_1, ...

It is manifest that, if y, $\frac{dy}{dx}$, ... are such that

$$u = \int_{x_0}^{x_1} f\left(x,\ y,\ \frac{dy}{dx},\ \ldots\right)$$

is a minimum, then

$$\int_{x_0}^{x_1} \left\{ N(y - \delta y)\,\delta y + P_1(y - \delta y)\,\frac{d\delta y}{dx} + \ldots + P_n(y - \delta y)\,\frac{d^n\delta y}{dx^n} \right\} dx \text{ is } -,$$

and

$$\int_{x_0}^{x_1} \left\{ N(y + \delta y)\,\delta y + P_1(y + \delta y)\,\frac{d\delta y}{dx} + \ldots + P_n(y + \delta y)\,\frac{d^n\delta y}{dx^n} \right\} dx \text{ is } +,$$

whatever value δy, $\frac{d\delta y}{dx}$, ... have initially, provided they continuously change from zero, and that δy, $\frac{d\delta y}{dx}$, ... as functions of x, are not truly discontinuous, it being supposed that y, $\frac{dy}{dx}$, ... respectively are not truly discontinuous.

In like manner, when u is a maximum,

$$\int_{x_0}^{x_1} \{N(y - \delta y)\,\delta y + \ldots\}\, dx \text{ is } +,$$

$$\int_{x_0}^{x_1} \{N(y + \delta y)\,\delta y + \ldots\}\, dx \text{ is } -.$$

So, excluding all cases wherein any of the functions

$$N,\ P_1,\ \ldots\ P_n,\ \frac{dN}{dy},\ \frac{dN}{dp_1},\ \ldots\ \frac{dN}{dp_n},\ \frac{dP_1}{dp_1},\ \frac{dP_1}{dp_2},\ \ldots\ \frac{dP_n}{dp_n},$$

are truly discontinuous for a value of x between x_0 and x_1 inclusive, (which cases should always be separately examined for themselves), it appears that, whenever, δu vanishing,

$$\delta^2 u = \int_{x_0}^{x_1} \left(\frac{d^2 f}{dy^2} \delta^2 y + 2 \frac{d^2 f}{dy dp_1} \delta y \delta p_1 + 2 \frac{d^2 f}{dy dp_2} \delta y \delta p_2 + \ldots + \frac{d^2 f}{dp_n^2} \delta p_n^2 \right) dx$$

is always positive, and can never be made to vanish, u is a minimum; and whenever it is always negative, and can never be made to vanish, u is a maximum: if δy, $\delta p_1, \ldots$ can be so chosen that its sign is not invariable, u is neither a maximum nor a minimum: in the case wherein it can vanish, although it cannot change sign, the first of the expressions $\int_{x_0}^{x_1} \left(\frac{d^3 f}{dy^3} \delta y^3 + \ldots \right) dx$, $\int_{x_0}^{x_1} \left(\frac{d^4 f}{dy^4} \delta y^4 + \ldots \right) dx \ldots$, which does not vanish must be examined, provided no truly discontinuous functions occur.

We may suppose that δy, $\delta p_1, \ldots \delta p_n$ vanish at the limits. For cases in which this is not requisite may be discussed by first supposing them to vanish at the limits, and then determining, by the method of the Differential Calculus, which of the solutions so obtained gives a maximum or minimum.

Now, when $\frac{\delta^2 f}{dp_n^2}$ is invariable in sign for all values of x between x_0' and x_1', by choosing $\frac{d^n \delta y}{dx^n} = \Big|^{a=\infty} \delta c \sin ax$, between those limits, and equal zero for all other values of x, and at the limits (which can be readily done, as has been shown, by Fourier's integrals), we secure a variation in which $\int_{x_0}^{x_1} \delta f(y + \delta y)\, dx$ has initially the same sign as $\int_{x_0'}^{x_1'} \frac{d^2 f}{dp_n^2} \delta p^2 dx$, or as $\frac{d^2 f}{dp_n^2}$. So when u is a minimum $\frac{d^2 f}{dp_n^2}$ must be always positive, and, when u is a maximum, always negative.

129. Again, whatever the form of δp_n be, the condition that it vanish at the limits enables us to express it in a series of the form

$$C_1 \sin\theta + C_2 \sin 2\theta + \ldots\ldots + C_r \sin r\theta + \ldots\ldots,$$

where $\theta = \dfrac{\pi(x-x_0)}{x_1-x_0}$.

This will give

$$\frac{\pi\delta p_{n-1}}{x_1-x_0} = C_0 - C_1 \cos\theta - \frac{1}{2} C_2 \cos 2\theta - \ldots - \frac{1}{r} C_r \cos r\theta - \ldots,$$

the condition that it vanish at the limits, showing that

$$C_0 - \frac{1}{2} C_2 - \frac{1}{4} C_4 - \ldots = 0,$$

$$C_1 + \frac{1}{3} C_3 + \frac{1}{5} C_5 + \ldots = 0;$$

and again

$$\frac{\pi^2\delta p_{n-2}}{(x_1-x_0)^2} = - C_1 \sin\theta - \frac{1}{2^2} C_2 \sin 2\theta - \ldots\ldots - \frac{1}{r^2} C_r \sin r\theta - \ldots,$$

the condition of vanishing at the limits showing that $C_0 = 0$. Now, as δy, δp_1, ... δp_n are finite for all concerned values of x, none of the coefficients C_0, C_1, ... C_r, ... can be infinite. For if C_r can be infinite $\int_{x_0}^{x_1} \delta p_n \sin \dfrac{r\pi(x-x_0)}{x_1-x_0}\, dx$ can be so. This cannot be, for $\int_{x_0}^{x_1} \sqrt{\delta p_n^2}\, dx$ is finite.

So, if $(x_1 - x_0)$ be small enough,

$$\delta^2 u = \int_{x_0}^{x_1} \left(\frac{d^2 f}{dy^2}\delta y^2 + 2\frac{d^2 f}{dy dp_1}\delta y \delta p_1 + \ldots + \frac{d^2 f}{dp_n^2}\delta p_n^2\right) dx$$

must have the same sign as $\int_{x_0}^{x_1} \dfrac{d^2 f}{dp_n^2}\delta p_n^2 dx$.

Therefore the *requisite* condition that $\dfrac{d^2 f}{dp_n^2}$ do not change sign is also *sufficient* when $(x_1 - x_0)$ is small enough.

130. Also, as long as (x_1-x_0) is so small that the invariability of sign in $\frac{d^2f}{dp_n^2}$ is a sufficient condition, the minimum (or maximum) values of $\delta^2u=\int_{x_0}^{x_1}\left(\frac{d^2f}{dy^2}\delta y^2+\ldots\ldots+\frac{d^2f}{dp_n^2}\delta p_n^2\right)dx$, for which $\frac{d\delta p_n}{dx}$ vanishes initially (vide Art. 125), obtained by varying δy, δp_1, ... δp_n, will, one be the value 0, given by $\delta y=\delta p_1=\ldots=\delta p_n=0$, and all the others will make δ^2u positive when u is a minimum, and negative when u is a maximum. These values can be obtained in the usual way. So for the least value of (x_1-x_0) for which it is possible that δ^2u vanish, some of these maxima or minima values will show what forms other than zero to assign to δy, δp_1, ... δp_n, so as to make δ^2u vanish. If (x_1-x_0) have a greater value than this, falsely discontinuous values can be generally assigned for δy, δp_1, ... δp_n so as to render δ^2u of a different sign from $\frac{d^2f}{dp_n^2}$, in those cases when the sign of δ^2u can be changed by a proper choice of functions for δy, δp_1, ... δp_n.

131. In individual cases the process of investigation can be much shortened by choosing

$$\delta^2u=\delta^2u+\int_{x_0}^{x_1}\frac{dF}{dx}dx,$$

where F is a function of x, y, p_1, ... p_{n-1}, and vanishes at the limits, and by observing that the differential equations to which the investigation of the minima and maxima of

$$\int_{x_0}^{x_1}\left(\frac{d^2f}{dy^2}\delta y^2+\ldots+\frac{d^2f}{dp_n^2}\delta p_n^2\right)dx$$

leads must be connected in character with those given by investigating the minima and maxima values of $\int_{x_0}^{x_1}f\,dx$.

132. Observe that the assumption of forms for δy, δp_1, ... δp_n, in series of sines or cosines of multiples of $\frac{\pi(x-x_0)}{x_1-x_0}$,

ought to enable us, when $\frac{d^2f}{dy^2}$, $\frac{d^2f}{dy\,dp_1}$, ... $\frac{d^2f}{dp_n^2}$ are assigned functions of x, to determine the greatest value of (x_1-x_0) for which δ^2u is invariable in sign without previously integrating the differential equation to which the general method would lead. The objection to such a method of investigation being simply that the analytical difficulties which would present themselves are generally greater than the difficulties to be encountered in integrating the differential equation. In fact, by such assumption, every problem in Calculus of Variations could be solved by solving, by Differential Calculus methods, a maxima and minima problem, in which the number of independent variables is infinite.

133. The process indicated in Art. 132 may be sometimes shortened by the following considerations. If δ^2u is always of the same sign for all values of C_1, C_2, ... C_r, ... (Art. 129), then C_1, C_2, ... C_r, ... can be so chosen as to render δ^2u a maximum or a minimum according as its sign is negative or positive. It follows that

$$\frac{d^2\delta^2u}{dC_r^2}$$

must be invariable in sign for all positive integral values of r. The application of this method will be more readily perceived by attention to a few examples.

134. For example let us consider (a being +),

$$u=\int_{x_0}^{x_1}(p^2-a^2y^2)\,dx.$$

The variations $\delta y=\delta c$, $\delta p=0$, decrease, and the variations $\delta y=0$, $\delta p=\delta c\sin\infty x$, increase u. But if we suppose y fixed at one limit, the first of these variations cannot be applied. We proceed to investigate under what circumstances u will admit of a minimum value when $y=0$ if $x=x_0$.

We have $\qquad N\equiv-2a^2y;\quad P\equiv 2p.$

So that $\qquad 2p+2a^2\frac{d^{-1}y}{dx^{-1}}=I(x).$

There is also the equation $\left|_{x_0}^{x_1} \frac{d^{-1}y}{dx^{-1}} \delta y = 0\right.$ to be satisfied.

If we introduce the restriction $\int_{x_0}^{x_1} pdx =$ a constant, *i.e.* if we suppose y fixed at both limits, this second equation is thereby sufficiently satisfied, and then the first equation will give

$$p + a^2 \int_{x_0}^{x} ydx + C = \tfrac{1}{2} I(x).$$

C must be determined by the equation, $p + C = 0$, for some value of x indefinitely near x_0, for which $I(x)$ vanishes.

This equation is solved by the system

$$\frac{dy}{dx} = Aa \cos a(x - x_0) + \tfrac{1}{2} I(x),$$

$$y = A \sin a(x - x_0),$$

$$Aa + C = 0,$$

and A is determined by requiring $A \sin a(x_1 - x_0)$ to have a given value.

Now consider $\delta^2 u = \int_{x_0}^{x_1} (\delta p^2 - a^2 \delta y^2)\, dx.$

This (Art. 128) will be positive if $(x_1 - x_0)$ is small enough, and our solution will give a minimum.

Now consider the minima values of $\delta^2 u$, supposing $\frac{d\delta\delta p}{dx}$ to be initially evanescent, and $(x_1 - x_0)$ small enough, they will require (Art. 125),

$$\delta p = \{A' a \cos a(x - x_0) + \tfrac{1}{2} I_\infty(x)\}\, \delta c,$$

$$\delta y = \{A' \sin a(x - x_0) + I_1(x)\}\, \delta c,$$

(vide Art. 135), and give $\frac{1}{\delta c}\int_{x_0}^{x_1} (\delta p^2 - a^2 \delta y^2)\, dx$ infinite, except when

$\delta y = 0$, $\delta p = 0$. This will be the case as long as $(x_1 - x_0) < \frac{\pi}{a}$. But if $(x_1 - x_0) = \frac{\pi}{a}$ we can assume

$$\delta p = A'a\,\delta c \cos a(x - x_0), \quad \delta y = A'\delta c \sin a(x - x_0),$$

and so obtain $\int_{x_0}^{x_1} (\delta p^2 - a^2 \delta y^2)\,dx$ evanescent for other values of δp, δy than zero, and if $(x_1 - x_0) > \frac{\pi}{a}$, by choosing

$$\delta p = A'a\delta c \cos a(x - x_0), \text{ from } x = x_0 \text{ till } x = x_0 + \frac{\pi}{2a},$$

$$\delta p = 0, \qquad \text{from } x = x_0 + \frac{\pi}{2a} \text{ till } x = x_1 - \frac{\pi}{2a},$$

$$\delta p = -A'a\delta c \cos a(x - x_0), \text{ from } x = x_1 - \frac{\pi}{2a} \text{ till } x = x_1,$$

δp having, if we choose, an instantaneous zero value both when $x = x_0$ and $x = x_1$ (for such permission would have no effect on the integral), and by choosing consistently,

$$\delta y = A'\delta c \sin a(x - x_0), \quad \delta c, \quad A'\delta c \sin a(x_1 - x),$$

between the same limits respectively, we secure that

$$\int_{x_0}^{x_1} (\delta p^2 - a^2\delta y^2)\,dx = -a^2\delta c^2\left(x_1 - \frac{\pi}{2a} - x_0 - \frac{\pi}{2a}\right) = -a^2\delta c^2\left(x_1 - x_0 - \frac{\pi}{a}\right),$$

so that the integral u does not admit of a minimum value.

135. *Idem aliter.* Expressing (Arts. 129, 132, 133) δy, δp in Fourier's series, thus,

$$\delta p = \frac{\pi}{x_1 - x_0}\left\{C_1 \cos\frac{\pi(x - x_0)}{x_1 - x_0} + \ldots + rC_r \cos\frac{r\pi(x - x_0)}{x_1 - x_0} + \ldots\right\},$$

$$\delta y = C_1 \sin\frac{\pi(x - x_0)}{x_1 - x_0} + \ldots + C_r \sin\frac{r\pi(x - x_0)}{x_1 - x_0} + \ldots,$$

we shall have

$$\delta^2 u = \int_{x_0}^{x_1} (\delta p^2 - a^2\delta y^2)\,dx = \frac{1}{2}\int_{x_0}^{x_1} \left\{\left(\frac{\pi}{x_1 - x_0}\right)^2 [\Sigma r^2 C_r^2] - a^2 \Sigma C_r^2\right\} dx,$$

not retaining terms that disappear after integration. $\delta^2 u$ will be positive; if always $\frac{\pi r}{x_1 - x_0} > a$. But if $\frac{\pi}{x_1 - x_0} < a$, a negative value can be readily obtained for it.

136. By $I_\infty(x)$ is meant a function, falsely discontinuous when finite, whose general value is zero, but which has other values for particular values of x, and some of these values infinite. It is obvious that, if (in 134) $I_1(\xi)$ is finite, $I_\infty(\xi)$ is infinite.

137. The investigation of Art. 134 can be exemplified geometrically as follows.

Let $OX_0 = x_0$, $OX_1 = x_1$, $OX' = x'$, X' being a point on X_0X_1 near X_1, and suppose $a(x_1 - x_0) < \frac{\pi}{2}$ (fig. 28). Trace a falsely discontinuous curve, with ordinate zero when $x = x_0$, and ordinate (A' being +),

$$\delta p = A'a \cos a(x - x_0),$$

from $x = x_0$ till $x = x'$, so that $X'P' = A'a \cos a(x' - x_0)$.

Bisect $X'X_1$ by an ordinate TP'', on which take a point P'', and draw through P' P'' the curve (A'' being +),

$$\delta p = A''a \cos a(x - B'),$$

and draw through $P''X_1$ the curve (A''' being +),

$$\delta p = A'''a \cos a(x - B'').$$

By taking TP'' large enough we can secure that

$$\int_{x_0}^{x_1} \delta p dx = 0.$$

Fig. 29 represents the curve which is the locus of

$$\int_{x_0}^{x} \delta p dx, \text{ or } \delta y.$$

We then have the following results,

$$X'P' = A'a \cos a(x' - x_0) = A''a \cos a(x' - B'');$$

$$TP'' = A''a \cos a\left(\frac{x' + x_1}{2} - B''\right) = A'''a \cos a\left(\frac{x' + x_1}{2} - B'''\right);$$

$$0 = A'''a \cos a(x_1 - B'''), \quad \text{or } B''' = \frac{\pi}{2a} + x_1.$$

The equations for δy will be,

for X_0Y', $\qquad \delta y = A' \sin a(x - x_0)$,

for $Y'Y''$,

$$\delta y = A' \sin a(x' - x_0) + A'' \sin a(x - B'') - A'' \sin a(x' - B''),$$

for $Y''X_1$, $\quad \delta y = A''' \sin a(x - B''') - A''' \sin a(x_1 - B''')$,

$$\text{or } \delta y = A' \sin a(x' - x_0) + A'' \sin a\left(\frac{x_1 + x'}{2} - B''\right) - A'' \sin a(x' - B'')$$

$$+ A''' \sin a(x - B''') - A''' \sin a\left(\frac{x_1 + x'}{2} - B''\right).$$

From these I find

$$A''' = A' \frac{\sin a\left(\dfrac{x_1 + 3x'}{4} - x_0\right)}{2 \sin a\left(\dfrac{x_1 - x'}{4}\right) \sin a\left(\dfrac{x_1 - x'}{2}\right)},$$

$$A'' \cos a(x' - B'') = A' \cos a(x' - x_0),$$

$$A'' \sin a(x' - B'') = A' \cos a(x' - x_0) \cot a\left(\frac{x_1 - x'}{2}\right) + A'''.$$

These results show that, for the curve X_0Y_1,

$$\delta p + a^2 \int_{x_0}^{x} \delta y dx - A'a = 0;$$

and, for the curve $Y'Y''$,

$$\delta p + a^2 \int_{x_0}^{x} \delta y dx - A'a$$

$$= a^2 (x - x') \{A' \sin a(x' - x_0) - A'' \sin a(x' - B'')\};$$

and, for the curve $Y''X_1$,

$$\delta p + a^2\int_{x_0}^{x}\delta y dx - A'a$$

$$= \frac{1}{2}a^2(x_1 - x')\,\{A'\sin a\,(x' - x_0) - A''\sin a\,(x' - B')\}$$

$$- A'''a\sin a\left(\frac{x_1 - x'}{2}\right) + \frac{1}{2}a^2(2x - x_1 - x')\,A''' - A'a.$$

Now suppose X' to move up to, and ultimately to coincide with, X_1, A' remaining constant. Then the ultimate value of $X'P'$ will be finite, and will be

$$A'a\cos a\,(x_1 - x_0).$$

Also ultimately,

$$A'''(x_1 - x')^2 = 4A'\sin a\,(x_1 - x_0),$$

$$A''(x_1 - x')^2 = A'''(x_1 - x')^2;$$

and we shall find that, if $\delta\delta p$ always $< \delta c$ (Art. 124),

$$\int_{x_0}^{x_1}\delta\delta p\,(\delta p + a^2\int_{x_0}^{x}\delta y dx - A'a)\,dx = \left(\int_{x_0}^{x'} + \int_{x'}^{x_1}\right)$$

$$= \int_{x'}^{x_1}\delta\delta p\,(\delta p + a^2\int_{x_0}^{x}\delta y dx - A'a)\,dx \text{ ultimately},$$

$$= \int_{x'}^{x_1}\delta\delta p\,.\,\delta p dx \text{ ultimately}.$$

If we might assume $\delta\delta p = f'(\delta y)$, this would not vanish, for $\Big|_{x'}^{x_1} f(\delta y)$ is generally finite. But such a variation would require $\dfrac{d\delta\delta p}{dx}$ to be infinite.

But if $\dfrac{d\delta\delta p}{dx}$ must be evanescent initially, as would be the case in considering $\delta p^2 - a^2\delta y^2$ to be a function of δy, δp, $\dfrac{d\delta p}{dx}$, as we do in Art. 130, in the equation

$$\int_{x'}^{x_1}\delta\delta p\delta p dx = \Big|_{x'}^{x_1}\delta\delta p\,.\,\delta y - \int_{x'}^{x_1}\frac{d\delta\delta p}{dx}\delta y dx,$$

$\delta\delta p$, vanishing when $x = x_1$, is ultimately less than any assignable quantity throughout the interval, $x = x'$ to $x = x_1$. And so the evanescence of

$$\int_{x'}^{x_1} \delta\delta p \,.\, \delta p dx$$

is secured.

This exemplifies our result of Art. 134, that

$$\delta p = \{A'a \cos a\,(x - x_0) + \tfrac{1}{2} I_2(x)\}\,\delta c,$$

$$\delta y = \{A' \sin a\,(x - x_0) + I_1(x)\}\,\delta c,$$

makes u a minimum when considered as a function of δy, δp, $\frac{d\delta p}{dx}$, although it does not make u a minimum when considered as a function of δy and δp.

138. In all the cases in which the method of Arts. 122 to 124 fails to be applicable,

$$f\left(x,\ y + \delta y,\ \frac{d(y + \delta y)}{dx},\ \ldots.\right)$$

fails to be expandible by Taylor's Theorem in powers of δy, $\frac{d\delta y}{dx}$, ... This is manifest.

Besides the cases in which Taylor's Theorem is recognized to fail in treatises on the Differential Calculus, that is besides the cases in which some of the differential coefficients of the functions concerned become infinite, we must add those cases in which any of the differential coefficients become truly discontinuous.

For instance, the function $\sqrt{(y + \delta y)^2}$ is ordinarily expandible according to powers of δy, the expansion being

$$\sqrt{y^2}\left(1 + \frac{\delta y}{y}\right).$$

But when y vanishes it is not so. For there is no series of ascending powers of δy, which can always be equated to $\sqrt{\delta y^2}$.

139. If we take as an example, a being $+$,

$$u = \int_{x_0}^{x_1} \{y^2 - 2a\sqrt{y^2(1+p^2)}\}\, dx,$$

since $$\frac{d\sqrt{y^2(1+p^2)}}{dy} = \sqrt{1+p^2}\,\frac{d\sqrt{y^2}}{dy} = \frac{y\sqrt{1+p^2}}{\sqrt{y^2}}$$

is truly discontinuous when $y=0$, Taylor's Theorem fails (Art. 138) to expand $\sqrt{(y+\delta y)^2}$ when $y=0$. So we cannot in this case express the variation δu in the usual manner. The proper expression for δu, when y vanishes, will be

$$\delta u = \int_{x_0}^{x_1} (-2a\sqrt{1+p^2}\,.\,\sqrt{\delta y^2})\, dx,$$

and is negative; so, $y=0$ renders u a maximum. If any other value of y, which does not change sign, render u a maximum or minimum, it must also render

$$\int_{x_0}^{x_1} (y^2 - 2ay\sqrt{1+p^2})\, dx$$

a maximum or minimum. We shall consider this integral in the next Article but one.

140. In connexion with the preceding investigation we observe that, δy, ... and their variations vanishing at the limits,

$$\delta^2 u = \delta \int_{x_0}^{x_1} (P_n - \frac{d^{-1}P_{n-1}}{dx^{-1}} + \ldots)\,\frac{d^n\delta y}{dx^n}\, dx$$

$$= \int_{x_0}^{x_1} \frac{d^n\delta y}{dx^n}\,\delta\left(P_n - \frac{d^{-1}P_{n-1}}{dx^{-1}}\right) dx + \int_{x_0}^{x_1} \frac{d^n\delta\delta y}{dx^n}\,(P_n - \frac{d^{-1}P_{n-1}}{dx^{-1}} + \ldots)\, dx.$$

Now the same conditions as render δu evanescent, viz. that

$$\int_{x_0}^{x_1} \frac{d^n\delta y}{dx^n}\,(P_n - \frac{d^{-1}P_{n-1}}{dx^{-1}} + \ldots)\, dx = 0,$$

when δy, δp, ... vanish at the limits, suffice to render evanescent

$$\int_{x_0}^{x_1} \frac{d^n\delta\delta y}{dx^n}\left(P_n - \frac{d^{-1}P_{n-1}}{dx^{-1}} + \dots\right),$$

under the same circumstances.

So $\delta^2 u$ will vanish if

$$\delta\left(P_n - \frac{d^{-1}P_{n-1}}{dx^{-1}} + \dots\right)$$

vanishes. Now if $\delta y'$, $\frac{d\delta y'}{dx}$, ... $\frac{d^n\delta y'}{dx^n}$, vanishing at the limits can have value other than zero, continuously changing from zero, and yet the equation

$$P_n - \frac{d^{-1}P_{n-1}}{dx^{-1}} + \dots = I(x)$$

be satisfied both by $y = y'$, ... and $y = y' + \delta y'$, ... then we shall have

$$\delta\left(P_n - \frac{d^{-1}P_{n-1}}{dx^{-1}} + \dots\right) = 0,$$

and so if $(x_1 - x_0)$ be so large that the equation

$$P_n - \frac{d^{-1}P_{n-1}}{dx^{-1}} + \dots = I(x)$$

admits of a solution in which y, $\frac{dy}{dx}$, ... remaining constant at the limits can nevertheless continuously change from one value to another for the same value of x in the interim, then $\delta^2 u$ will vanish. Under these circumstances then the test $\delta u = 0$ will fail to establish the existence of maxima or minima. The problem which we proceed to discuss will be found to exemplify the theory of this Article.

141. Let (a being +),

$$u = \int_{x_0}^{x_1}(y^2 - 2ay\sqrt{1+p^2})\,dx,$$

(confer Arts. 96—100, and Appendix)............(1).

This gives

$$\delta u = \int_{x_0}^{x_1}\left(2y\delta y - 2a\sqrt{1+p^2}\,\delta y - \frac{2apy\delta p}{\sqrt{1+p^2}}\right)dx \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$\delta^2 u = \int_{x_0}^{x_1}\{2\delta y^2 - \frac{4ap}{\sqrt{1+p^2}}\,\delta y\delta p - \frac{2ay}{(1+p^2)^{\frac{3}{2}}}\,\delta p^2\}\,dx \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

If y change sign, or y and p vanish together, there can be neither a maximum nor a minimum value of u, and if y is positive, there cannot be a minimum.

We commence with the discussion of the circumstances under which δu vanishes. Suppose, if possible, y fixed at the limits and always positive; then, if $(x_1 - x_0)$ is small enough, u can have a maximum value. We shall (chiefly) consider the case in which y, though not vanishing, is indefinitely small both when $x = x_0$, and when $x = x_1$; but the results of our analysis will not be confined to cases so restricted.

We obtain the following equations whenever $\delta u = 0$,

$$\lambda - \frac{2apy}{\sqrt{1+p^2}} - 2\int_{x_0}^{x}(y - a\sqrt{1+p^2})\,dx = I(x) \ \ldots\ldots\ldots(4),$$

$$-\frac{2ay}{(1+p^2)^{\frac{3}{2}}}\frac{dp}{dx} - \frac{2ap^2}{\sqrt{1+p^2}} - 2y + 2a\sqrt{1+p^2} = \frac{dI}{dx} \ \ldots\ldots\ldots\ldots(5),$$

$$-\frac{2apy}{(1+p^2)^{\frac{3}{2}}}\frac{dp}{dx} + \frac{2ap}{\sqrt{1+p^2}} - 2py = p\frac{dI}{dx} \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

$$\frac{2ay}{\sqrt{1+p^2}} - y^2 = \int_{x_0}^{x} p\frac{dI}{dx}\,dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

$\int_{x_0}^{x} p\frac{dI}{dx}\,dx$ may change value either when p is infinite or when $\frac{dI}{dx}$ is infinite. Thus, if $\Big|^{x+\delta x} p$ and $\Big|^{x} p$ differ by less than any assignable quantity, and $\Big|^{x+\delta x.\cos^2\theta} p$ is always intermediate to them,

$$\int_x^{x+\delta x} p\frac{dI}{dx}\,dx = \Big|^x p \,.\, \Big|_x^{x+\delta x} I.$$

Now our equation,

$$\lambda - \frac{2apy}{\sqrt{1+p^2}} - 2\int_{x_0}^x (y - a\sqrt{1+p^2})\,dx = I(x),$$

is not satisfied by $y=0$; this solution was introduced when we multiplied by p.

Nor is it, between any values for x, satisfied by the solution of

$$y^2 - \frac{2ay}{\sqrt{1+p^2}} = b^2,$$

given by $p=0$, $y = a + \sqrt{a^2+b^2}$, unless $b^2 = -a^2$. For for this solution $\int_{x_0}^x (y - a\sqrt{1+p^2})\,dx$ cannot be equated to the other terms of the equation.

Nor is it satisfied by any solution which requires (confer Art. 96) p to be truly discontinuous and finite, y and p^2 being only falsely discontinuous. Such solutions render $\frac{2apy}{\sqrt{1+p^2}}$ truly discontinuous when y does not vanish, and no other terms in our equation.

But our equation is satisfied by supposing, for any value, x', of x,

$$p = \infty, \int_{x'-\delta x}^{x'+\delta x} \sqrt{1+p^2}\,dx = \Big|_{x'-\delta x}^{x'+\delta x} y,$$

even although $I(x')$ vanish. Now, through any interval throughout which p is finite, $\int p\,dx$, or y, cannot, when expressed in terms of x, contain any term of the form $I_1(x)$; and therefore, throughout such interval, $I(x)$ can only affect the value of p, and cannot affect that of y. That is we must have

$$y^2 - \frac{2ay}{\sqrt{1+(p-I)^2}} = b^2,$$

and, as far as y is concerned, we may put $y = \eta$, where

$$\eta^2 - \frac{2a\eta}{\sqrt{1+\frac{d\eta^2}{dx}}} = b^2.$$

Confining our attention to cases in which $I(x)$ vanishes except when $x = x_0$, and $x = x_1$, let P_0P_1 be a curve satisfying the equation (fig. 30)

$$y^2 - \frac{2ay}{\sqrt{1+\frac{dy^2}{dx}}} = b^2,\ b^2 +,$$

δu will vanish for the curve $X_0P_0P_1X_1$. The equation to such a curve could be appropriately expressed by assuming for y a series of sines of multiples of $\frac{\pi(x-x_0)}{x_1-x_0}$, which should, as shown in our discussion of Fourier's Theorem, have for value $x \tan P_0'X_0X_1$, from $x = x_0$ till $x =$ abscissa of P_0', then the ordinate of $P_0'P_1'$, and then $(x_1 - x)\tan P_1'X_1X_0$, from $x =$ abscissa of P_1' till $x = x_1$, and then make P_0', P_1', move up to, and ultimately coincide with, P_0, P_1, respectively.

For the curve $X_0P_0P_1X_1$ we have, $y_0 = P_0X_0$, $y_1 = P_1X_1$, &c., taking a case when p is positive for $x = x_0$, and negative for $x = x_1$;

$$\left.\begin{aligned}
&\lambda = I(x_0);\\
&\lambda - 2ay_0 \sin P_0'P_0X_0 + 2ay_0 = 0;\\
&\lambda + 2ay_1 \sin P_1'P_1X_1 - 2\,(\text{area } P_0X_0P_1X_1) + 2aX_0P_0P_1 = 0;\\
&\lambda - 2\,(\text{area } P_0X_0P_1X_1) + 2aX_0P_0P_1 + 2aP_1X_1 = I(x_1);\\
&\text{so that}\\
&I(x_0) = 2ay_0(\sin P_0'P_0X_0 - 1);\ I(x_1) = 2ay_1(1 - \sin P_1'P_1X_1).
\end{aligned}\right\}\ (8)$$

Observe that the equation,

$$2ay_0 - 2ay_0 \sin P_0'P_0X_0 = 2ay_0 + 2aP_0P_1 - 2\,(\text{area } P_0X_0X_1P_1) + 2ay_1 \sin P_1'P_1X_1,$$

is satisfied by the curve.

In the curve $X_0P_0P_1X_1$ the initial value of $\int_{x_0}^{x} p\frac{dI}{dx}\,dx$ is $-y^2$: this continues during an infinitely small change in the value of x, viz., till $y = y_0$, when the value of $\int_{x_0}^{x} p\frac{dI}{dx}\,dx$ becomes $\frac{2ay_0}{\sqrt{1+p^2}} - y_0^2$, ($b^2$ is positive for hyperbolic constructors,) and remains so also during an infinitely small change in the value of x, viz., till

$$y_0^2 - \frac{2ay_0}{\sqrt{1+p^2}} = b^2.$$

After this the value of $\int_{x_0}^{x} p\frac{dI}{dx}\,dx$ remains $= -b^2$ till the neighbourhood of $x = x_1$ is reached, when it changes in value in a manner precisely like the manner in which it changes in the neighbourhood of $x = x_0$. The values of $I(x_0)$, $I(x_1)$, when b^2 is negative, can be found in a manner precisely similar.

We have now to consider the sign of $\delta^2 u$, and the circumstances under which it can vanish. The investigation will be facilitated by slight transformations. Thus

$$\delta^2 u = \int_{x_0}^{x_1} \left\{2\delta y^2 - \frac{4ap}{\sqrt{1+p^2}}\,\delta y\delta p - \frac{2ay}{(1+p^2)^{\frac{3}{2}}}\,\delta p^2\right\} dx \;\ldots\ldots\; (3),$$

$$= \int_{x_0}^{x_1} \delta p \left\{-\frac{2ap\delta y}{\sqrt{1+p^2}} - 2\int_{x_0}^{x} \delta y dx - \frac{2ay}{(1+p^2)^{\frac{3}{2}}}\,\delta p + 2a\int_{x_0}^{x} \frac{p\delta p}{(1+p^2)^{\frac{1}{2}}}\,dx\right\} dx$$

$$+ \int_{x_0}^{x_1} \frac{d}{dx}\left[\left\{2\int_{x_0}^{x} \delta y dx - 2a\int_{x_0}^{x} \frac{p\delta p}{(1+p^2)^{\frac{3}{2}}}\,dx\right\} \delta y\right] dx \;\ldots\ldots\; (9),$$

$$= \int_{x_0}^{x_1} \delta p \left[\int_{x_0}^{x} \left\{-\frac{2a\delta y}{(1+p^2)^{\frac{3}{2}}}\frac{dp}{dx} - 2\delta y\right\} dx - \frac{2ay}{(1+p^2)^{\frac{3}{2}}}\,\delta p\right] dx \ldots\ldots (10),$$

$$=\int_{x_0}^{x_1}\delta p\left\{\int_{x_0}^{x}\delta y\left(-\frac{2a}{y\sqrt{1+p^2}}+\frac{1}{y}\frac{dI}{dx}\right)dx-\frac{2ay}{(1+p^2)^{\frac{3}{2}}}\,\delta p\right\}dx\ldots\ldots(11),$$

$$=\Big|_{x_0}^{x_1}\left\{\delta y\int_{x_0}^{x}\left(-\frac{2a}{y\sqrt{1+p^2}}+\frac{1}{y}\frac{dI}{dx}\right)\delta y\,dx\right\}$$

$$+\int_{x_0}^{x_1}\left\{\left(\frac{2a}{y\sqrt{1+p^2}}-\frac{1}{y}\frac{dI}{dx}\right)\delta y^2-\frac{2ay}{(1+p^2)^{\frac{3}{2}}}\,\delta p^2\right\}dx\ldots\ldots(12),$$

$$=\int_{x_0}^{x_1}\left\{\left(\frac{2a}{y\sqrt{1+p^2}}-\frac{1}{y}\frac{dI}{dx}\right)\delta y^2-\frac{2ay}{(1+p^2)^{\frac{3}{2}}}\,\delta p^2\right\}dx\ldots\ldots(13).$$

δy vanishing absolutely at both limits, $\dfrac{\delta y}{y}$ vanishes also, and so the appearance of y in the denominator does not affect the process, as y is supposed never actually to vanish.

(13) must be negative, if (x_1-x_0) is small enough. We have now, in accordance with our method, to make (13) a maximum, δy, δp, $\dfrac{d\delta p}{dx}$ being considered the quantities subject to variation, and y, p, being known functions of x satisfying (4). So that

$$-\frac{d^{-1}}{dx^{-1}}\left\{\frac{4ay\delta p}{(1+p^2)^{\frac{3}{2}}}\right\}-\frac{d^{-2}}{dx^{-2}}\left(\frac{4a\delta y}{y\sqrt{1+p^2}}-\frac{2}{y}\frac{dI}{dx}\delta y\right)=I'(x)\ldots\ldots(14).$$

Had we considered (13) as a function of δy, δp, only, we should have had,

$$-\frac{4ay\delta p}{(1+p^2)^{\frac{3}{2}}}-\frac{d^{-1}}{dx^{-1}}\left(\frac{4a\delta y}{y\sqrt{1+p^2}}-\frac{2}{y}\frac{dI}{dx}\delta y\right)=I''(x)\ldots\ldots(15),$$

the result which (14) gives when $I'(x)$ is always zero. Now this is the relation which holds between δy, δp, when a passage is made from one curve to another curve of the series (7), a result which the Theory of Art. 140 enables us to foresee, but which we shall proceed independently to establish.

For such a passage requires that (5)

$$\delta\frac{dI}{dx} = -\frac{2ay}{(1+p^2)^{\frac{3}{2}}}\frac{d\delta p}{dx} + \frac{6apy}{(1+p^2)^{\frac{5}{2}}}\frac{dp}{dx}\delta p$$

$$-\frac{2a}{(1+p^2)^{\frac{3}{2}}}\frac{dp}{dx}\delta y - \frac{2ap}{(1+p^2)^{\frac{3}{2}}}\delta p - 2\delta y,$$

and we have

$$\frac{1}{2}\frac{dI''(x)}{dx} = -\frac{2ay}{(1+p^2)^{\frac{3}{2}}}\frac{d\delta p}{dx} + \frac{6apy}{(1+p^2)^{\frac{5}{2}}}\frac{dp}{dx}\delta p - \frac{2ap}{(1+p^2)^{\frac{3}{2}}}\delta p$$

$$-\frac{2a\delta y}{y\sqrt{1+p^2}} + \frac{1}{y}\frac{dI}{dx}\delta y = -\frac{2ay}{(1+p^2)^{\frac{3}{2}}}\frac{d\delta p}{dx} + \frac{6apy}{(1+p^2)^{\frac{5}{2}}}\frac{dp}{dx}\delta p$$

$$-\frac{2ap}{(1+p^2)^{\frac{3}{2}}}\delta p - \frac{2a}{(1+p^2)^{\frac{3}{2}}}\frac{dp}{dx}\delta y - 2\delta y.$$

Hence, when $(x_1 - x_0)$ is sufficiently small, $\delta^2 u$ is made a maximum by so choosing δy, δp, as to pass from one curve to another of the series, (14). And, when $(x_1 - x_0)$ has the value for which some of these maxima vanish, we come to that value beyond which it may be possible so to choose δy, δp, that $\delta^2 u$ may be positive.

So, always when $y_0 = y_1 > a$, and $(x_1 - x_0) < 2a$, our solutions given by $\delta u = 0$ make u a maximum. The question, which among these solutions gives u the greatest value, should properly be investigated by the methods of the Differential Calculus. When (99) the locus of ultimate intersections is met u will not be a maximum. If $(x_1 - x_0) > 2a$ and $< 4a$, y can be so chosen as to make u infinite; this can be done by making y constant for values of x not indefinitely near x_0, x_1, and then making this constant value infinite. We so obtain

$$u = \int_{x_0+}^{x_1-} (y^2 - 2ay\sqrt{1+p^2})\,dx - 2ay^2,$$

$$= (y^2 - 2ay)(x_1 - x_0) - 2ay^2,$$

which may be made infinitely great when $(x_1 - x_0) > 2a$. It will be readily perceived that no solution renders u a maximum when $(x_1 - x_0) > 4a$.

142. As an illustration, suppose (fig. 31)

$$X_0 B_0 = b_0, \quad X_0 X_1 = x_1 - x_0,$$

where
$$x_1 - x_0 = 2\int_{b_0}^{a+\sqrt{a^2+b_0^2}} \frac{y^2 - b_0^2}{\sqrt{4a^2y^2 - (y^2 - b_0^2)}}\, dy,$$

and is in value intermediate to $2a$ and $4a$.

143. Take $X_0P_0 > b_0$, and let $X_0P_0 = b$. Draw the curve $P_0Q_0Q_1P_1$ in which Q_0Q_1 is a straight line parallel to X_0X_1, and P_0Q_0, Q_1P_1, form each a solution, from $p = \infty$ to $p = 0$, and from $p = 0$ to $p = -\infty$ respectively, of

$$\frac{2ay}{\sqrt{1+p^2}} = y^2 - b^2.$$

Then for $P_0Q_0Q_1P$ we shall have

$$Q_0Q_1 = x_1 - x_0 - 2\int_b^{a+\sqrt{a^2+b^2}} \frac{y^2 - b^2}{\sqrt{4a^2y^2 - (y^2 - b^2)^2}}\, dy,$$

$$\text{for } p = \frac{\sqrt{4a^2y^2 - (y^2 - b^2)^2}}{y^2 - b^2},$$

$$u = (a + \sqrt{a^2 + b^2})^2\, Q_0Q_1 - 2a\,(a + \sqrt{a^2 + b^2})\, Q_0Q_1 - 2ab^2$$
$$+ 2\int_b^{a+\sqrt{a^2+b^2}} \frac{y^2\,(y^2 - b^2) - 4a^2y^2}{\sqrt{4a^2y^2 - (y^2 - b^2)^2}}\, dy$$
$$= b^2\, Q_0Q_1 - 2ab^2 + \int_b^{a+\sqrt{a^2+b^2}} \frac{2b^2\,(y^2 - b^2)\, dy}{\sqrt{4a^2y^2 - (y^2 - b^2)^2}}$$
$$- 2\int_b^{a+\sqrt{a^2+b^2}} \sqrt{4a^2y^2 - (y^2 - b^2)^2}\, dy$$
$$= b^2\,(x_1 - x_0 - 2a) - 2\int_b^{a+\sqrt{a^2+b^2}} \sqrt{4a^2y^2 - (y^2 - b^2)^2}\, dy.$$

So that

$$\frac{du}{db} = 2b(-2a + x_1 - x_0) + 4ab - 2\int_b^{a+\sqrt{a^2+b^2}} \frac{2b\,(y^2 - b^2)}{\sqrt{4a^2y^2 - (y^2 - b^2)^2}}\,dy$$

$$= + 2b \,.\, Q_0Q_1,$$

which is positive, increasing as Q_0Q_1 increases.

So that $y =$ ordinate of $X_0B_0B_1X_1$ gives no maximum. It should be noticed, accordingly, that when $x_1 - x_0 = 4a$, the circle, $y^2 + \left(x - \frac{x_0 + x_1}{2}\right)^2 = 4a^2$, does not make u a maximum.

Of course the passage from one curve of the set $X_0P_0Q_0Q_1P_1X_1$ to the next requires δp infinite when $x = x_0$, and when $x = x_1$. But points on the curve, p_0, p_1, can be taken very near P_0, P_1, respectively, and connected with B_0, B_1, respectively by a curve, as in fig. 31, so that u for $X_0B_0p_0Q_0Q_1p_1B_1X_1$ differ from u for $X_0B_0P_0Q_0Q_1P_1B_1X_1$ by less than any finite quantity when $B_0P_0P_1B_1$ is finite, and yet the difference between the values of p in the two curves, $X_0B_0B_1X_1$, $X_0B_0p_0p_1B_1X_1$, for the same value of x, be always finite, and so the passage from one curve of the set $X_0B_0p_0Q_0Q_1p_1B_1X_1$ to the next will not require at any point δp to be otherwise than infinitely small.

144. In Art. 143 the existence of a variation, increasing u, is established in a particular case. It will be readily observed that the possibility of this might have been foreseen, inasmuch as the curve, in its course, touches the locus of the ultimate intersections, and so $\delta^2 u$ has not invariable sign. In like manner it will not be without interest to show that (fig. 36), if in the curve P_0P_1 $\frac{d^2y}{dx^2}$ vanish both at P_0 and P_1, the curve being a portion concave throughout to the axis of x, of the curve traced by an elliptic constructor, a variation can be assigned so as to give a curve through P_0P_1 in which u is increased. Thus, if $X_0P_0 = \beta_0$, and β be not much less than β_0, Art. 100 shows that the curve corresponding to β will intersect X_0P_0. Let that portion of it, $\Pi_0\Pi'$, which is nearer X_0 than X_1, be moved parallel to Ox till it pass through P_0, and the portion nearer X_1 than X_0 be moved

till it pass through P_1, and the continuity be maintained by a line joining the ends, and therefore parallel to Ox. We shall have for P_0P_1,

$$u_0 = -2a\beta_0^2 - 2\int_{\beta_0}^{a+\sqrt{a^2-\beta_0^2}} \frac{4a^2y^2 - y^2(y^2+\beta_0^2)}{\sqrt{4a^2y^2 - (y^2+\beta_0^2)^2}}\,dy,$$

$$x_1 - x_0 = 2\int_{\beta_0}^{a+\sqrt{a^2-\beta_0^2}} \frac{y^2+\beta_0^2}{\sqrt{4a^2y^2 - (y^2+\beta_0^2)^2}}\,dy.$$

And if

$$\xi = 2\int_{\beta_0}^{a+\sqrt{a^2-\beta^2}} \frac{y^2+\beta^2}{\sqrt{4a^2y^2 - (y^2+\beta^2)^2}}\,dy,$$

ξ will be $< (x_1 - x_0)$, by Art. 100. For this curve

$$\begin{aligned} u &= -2a\beta_0^2 - \beta^2(x_1 - x_0 - \xi) \\ &\qquad - 2\int_{\beta_0}^{a+\sqrt{a^2-\beta^2}} \frac{4a^2y^2 - y^2(y^2+\beta^2)}{\sqrt{4a^2y^2 - (y^2+\beta^2)^2}}\,dy \\ &= -2a\beta_0^2 - \beta^2(x_1 - x_0 - \xi) - 2\int_{\beta_0}^{a+\sqrt{a^2-\beta^2}} \sqrt{4a^2y^2 - (y^2+\beta^2)^2}\,dy \\ &\qquad - 2\beta^2\int_{\beta_0}^{a+\sqrt{a^2-\beta^2}} \frac{(y^2+\beta^2)\,dy}{\sqrt{4a^2y^2 - (y^2+\beta^2)^2}} \\ &= -2a\beta_0^2 - \beta^2(x_1 - x_0) - 2\int_{\beta_0}^{a+\sqrt{a^2-\beta^2}} \sqrt{4a^2y^2 - (y^2+\beta^2)^2}\,dy, \end{aligned}$$

$$\begin{aligned} \frac{du}{d\beta} &= -2\beta(x_1 - x_0) + \int_{\beta_0}^{a+\sqrt{a^2-\beta^2}} \frac{4\beta(y^2+\beta^2)}{\sqrt{4a^2y^2 - (y^2+\beta^2)^2}}\,dy \\ &= -2\beta(x_1 - x_0 - \xi), \end{aligned}$$

so that u decreases as β increases, and therefore u_0 is not a maximum.

145. In like manner it can be shown that (fig. 37) a complete portion convex to the axis of x of a curve traced by an elliptic constructor will not give u a maximum.

We shall have for P_0P_1

$$u_0 = -2a\beta_0^2 - 2\int_{a-\sqrt{a^2-\beta_0^2}}^{\beta_0} \frac{4a^2y^2 - y^2(y^2+\beta_0^2)}{\sqrt{4a^2y^2-(y^2+\beta_0^2)^2}}\,dy,$$

$$x_1 - x_0 = 2\int_{a-\sqrt{a^2-\beta_0^2}}^{\beta_0} \frac{y^2+\beta_0^2}{\sqrt{4a^2y^2-(y^2+\beta_0^2)^2}}\,dy.$$

And if (compare Appendix on Art. 100, and fig. 34)

$$\xi = 2\int_{a-\sqrt{a^2-\beta^2}}^{\beta_0} \frac{y^2+\beta^2}{\sqrt{4a^2y^2-(y^2+\beta^2)^2}}\,dy,$$

where β is to be taken not much less than β_0, we shall have ξ less than $(x_1 - x_0)$. For

$$\int_{a-\sqrt{a^2-\beta^2}}^{\beta} \frac{(y^2+\beta^2)\,dy}{\sqrt{4a^2y^2-(y^2+\beta^2)^2}} = \int_0^{\frac{\pi}{2}} \sqrt{a^2\sin^2\psi + \beta^2\cos^2\psi}\,d\psi - \sqrt{a^2-\beta^2},$$

$$\frac{d}{d\beta}\int_{\beta}^{\beta_0} \frac{y^2+\beta^2}{\sqrt{4a^2y^2-(y^2+\beta^2)^2}}\,dy = -\frac{2\beta^2}{\sqrt{4a^2\beta^2-4\beta^4}}$$

$$+\int_{\beta}^{\beta_0} \frac{2\beta\{4a^2y^2-(y^2+\beta^2)^2\} + 2\beta(y^2+\beta^2)^2}{\{4a^2y^2-(y^2+\beta^2)^2\}^{\frac{3}{2}}}\,dy.$$

So that

$$\frac{d\xi}{d\beta} = \int_0^{\frac{\pi}{2}} \frac{2\beta\cos^2\psi\,d\psi}{\sqrt{a^2\sin^2\psi+\beta^2\cos^2\psi}} + \frac{2\beta}{\sqrt{a^2-\beta^2}} - \frac{2\beta}{\sqrt{a^2-\beta^2}}$$

$$+\int_{\beta}^{\beta_0} \frac{16a^2\beta y^2\,dy}{\{4a^2y^2-(y^2+\beta^2)^2\}^{\frac{3}{2}}},$$

showing that ξ increases with β, and is therefore greater when $\beta = \beta_0$ than when β is a little less than β_0, *i.e.* $(x_1 - x_0) > \xi$. If then we trace the curve corresponding to β, divide it in the point equidistant from X_0, X_1, and remove the pieces parallel Ox towards, and away from, Oy, till they pass respectively through

P_0, P_1, and join the broken ends by a straight line which will be parallel to Ox, we shall have for the curve so formed,

$$u = -2a\beta_0^2 - 2\int_{a-\sqrt{a^2-\beta^2}}^{\beta_0} \frac{4a^2y^2 - y^2(y^2+\beta^2)}{\sqrt{4a^2y^2 - (y^2+\beta^2)^2}}\,dy$$

$$+ (x_1 - x_0 - \xi)\{2a^2 - \beta^2 - 2a\sqrt{a^2-\beta^2} - 2a^2 + 2a\sqrt{a^2-\beta^2}\}$$

$$= -2a\beta_0^2 - (x_1 - x_0 - \xi)\,\beta^2 - \int_{a-\sqrt{a^2-\beta^2}}^{\beta_0} \frac{2\beta^2(y^2+\beta^2)}{\sqrt{4a^2y^2 - (y^2+\beta^2)^2}}\,dy$$

$$-2\int_{a-\sqrt{a^2-\beta^2}}^{\beta_0} \sqrt{4a^2y^2 - (y^2+\beta^2)^2}\,dy$$

$$= -2a\beta_0^2 - (x_1 - x_0)\,\beta^2 - 2\int_{a-\sqrt{a^2-\beta^2}}^{\beta_0} \sqrt{4a^2y^2 - (y^2+\beta^2)^2}\,dy,$$

$$\frac{du}{d\beta} = -2(x_1 - x_0)\,\beta + \int_{a-\sqrt{a^2-\beta^2}}^{\beta_0} \frac{4\beta(y^2+\beta^2)\,dy}{\sqrt{4a^2y^2 - (y^2+\beta^2)^2}}$$

$$= -2\beta(x_1 - x_0 - \xi),$$

so that, as before, u decreases as β increases, and therefore u_0 is not a maximum.

146. We will now consider the question whether, among the maxima whose existence we have established, there ever exists a maximorum maximum. For a maximum it is sufficient that no variation throughout which δp is initially evanescent shall make u greater. For a maximorum maximum the variations in which δp may be initially finite or infinitely great must also not make u greater, and the determination of maximorum maxima is a question properly belonging to the Differential Calculus, not to the Variations Calculus. In the example before us the maximorum maxima should be found by regarding u as a function of b^2, x_1, x_0, and the constant ξ_0 which so attaches to x, that, wherever x occurs, $x - \xi_0$ occurs in the integration of (7). We shall show when only $\dfrac{du}{db^2}$ can vanish. A maximum maximorum may then be given, if

u change sign properly, and may also be given for such values, if any, of b^2 that, for a bordering value of b^2, u is imaginary. We shall confine ourselves to the simplest case of symmetry, in which (fig. 32) $y_0 = y_1$, although much of what follows holds in the more general case. The reader will diminish the theoretical difficulties which attach to the following investigation by regarding y to be the ordinate of a curve $X_0 p_0 p_1 X_1$, wherein p_0 is so near to P_0, and p_1 to P_1, that no error arises from this difference, and then making p_0, p_1, move up to and ultimately coincide with P_0, P_1. It will be occasionally essential to mark the difference between X_0 and the foot of P_0's abscissa, *i.e.*, the ultimate position of the foot of p_0's abscissa. When this is necessary we shall denote the abscissa of X_0 by x_0, and of P_0, (or p_0's ultimate position,) by $x_0 +$. Similarly x_1 and $x_1 -$ will denote respectively the abscissæ of X_1 and P_1, or p_1's ultimate position.

We have

$$\frac{du}{db^2} = \int_{x_0}^{x_1} \left(2y\frac{dy}{db^2} - 2a\sqrt{1+p^2}\frac{dy}{db^2} - \frac{2apy}{\sqrt{1+p^2}}\frac{dp}{db^2}\right) dx$$

$$= -\Big|_{x_0}^{x_1} \frac{2apy}{\sqrt{1+p^2}}\frac{dy}{db^2} + \int_{x_0}^{x_1} \left[2\left(y - a\sqrt{1+p^2}\right)\frac{dy}{db^2}\right.$$

$$\left. + \left\{\frac{2ap^2}{\sqrt{1+p^2}} + \frac{2ay}{(1+p^2)^{\frac{3}{2}}}\frac{dp}{dx}\right\}\frac{dy}{db^2}\right] dx$$

$$= -\int_{x_0}^{x_1} \frac{dI}{dx} \cdot \frac{dy}{db^2}\, dx.$$

For y is indefinitely small, and $\frac{dy}{db^2}$ zero at both limits, y being then the ordinate of X_0, X_1 respectively.

Now $\frac{dI}{dx}$ vanishes except at the limits, when we have

$$\int_{x_0}^{x_0+} \frac{dI}{dx}\, dx = -I(x_0) = 2ay_0\left(1 - \frac{p_0}{\sqrt{1+p_0^{\,2}}}\right),$$

$$\int_{x_1-}^{x_1} \frac{dI}{dx}\, dx = I(x_1) = 2ay_1\left(1 + \frac{p_1}{\sqrt{1+p_1^{\,2}}}\right),$$

so that

$$\frac{du}{db^2} = -2ay_0\frac{dy_0}{db^2}\left(1 - \frac{p_0}{\sqrt{1+p_0^2}}\right) - 2ay_1\frac{dy_1}{db^2}\left(1 + \frac{p_1}{\sqrt{1+p_1^2}}\right).$$

Now y_0, y_1, increase with b^2, except, on the one hand, between the locus of the ultimate intersections, and the locus of the cusps, and on the other hand when $b^2 -$, between the locus of the points of contrary flexure and the locus of the ultimate intersections (vide figs. 14, 15, 34—37). So u would be a minimum among the maxima when y_0, y_1, are the ordinates of points on the locus of the ultimate intersections of curves for which $b^2 +$, did the invariability of sign in $\delta^2 u$ permit of its being a maximum. And likewise u would be a maximum among the maxima when y_0, y_1 are the ordinates of points on the locus of the ultimate intersections of curves for which $b^2 -$, were it not for the same reason. Be it observed that $\delta^2 u$ will also vanish for these curves.

In fig. 35 the locus of the ultimate intersections is separately traced for curves symmetrical with respect to the axis of y. The branches AQ, $A'Q'$, correspond to the branches in fig. 14, AP, $A'P'$, to those in fig. 15, and OP, OP' to those in fig. 34. We have

$$A\infty = \infty O = O\infty' = \infty' A' = a,$$

$$OD = OD' = \frac{\pi a}{2},$$

$$DP = D'P' = a.$$

We conclude that a curve making u a maximum can be drawn through points situated as B_0, B_1, but not through points situated as B_0', B_1'. Be it observed that, however small y_0, y_1 be, it is always possible to take $(x_1 - x_0)$ so small that a curve may be drawn joining the points (x_0, y_0) and (x_1, y_1), so that for it u is a maximum relatively to all curves through those points.

147. The following problem, in simple Maxima and Minima, is not without interest, owing to its resemblance to the investigation of the preceding Articles.

It is required to make a maximum, through the variation of y_1 (fig. 33),

$$u \equiv -ay_1^2 - ay_0^2 + \int_{x_0}^{x_1} (y^2 - 2ay \operatorname{cosec} \alpha)\, dx,$$

where $$\cot \alpha = \frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}.$$

I find

$$u \equiv -ay_1^2 - ay_0^2 + \frac{1}{3}(y_1^2 + y_1y_0 + y_0^2)(x_1 - x_0)$$
$$- a(y_1 + y_0)\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$

$$\frac{du}{dy_1} \equiv -2ay_1 + \frac{2y_1 + y_0}{3}(x_1 - x_0) - a\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$
$$- \frac{a(y_1^2 - y_0^2)}{\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}}.$$

If $y_1 = y_0$, $\dfrac{du}{dy_1} = -2ay_0 + (y_0 - a)(x_1 - x_0)$,

which is negative; but if $y_1 = 0$,

$$\frac{du}{dy_1} = \frac{1}{3}y_0(x_1 - x_0) - a\sqrt{(x_1 - x_0)^2 + y_0^2} + \frac{ay_0^2}{\sqrt{(x_1 - x_0)^2 + y_0^2}},$$

which is positive if

$$y_0^2(x_1 - x_0)^4 + y_0^4(x_1 - x_0)^2 > 9a^2(x_1 - x_0)^4,$$

or $$y_0^2 + \frac{(x_1 - x_0)^2}{2} > (x_1 - x_0)\sqrt{9a^2 + \frac{(x_1 - x_0)^2}{4}};$$

this can be always secured, if we are at liberty to take $(x_1 - x_0)$ small enough. This leads to the conclusion that we may always expect, in taking $(x_1 - x_0)$ small enough, to find that

$$\int_{x_0}^{x_1} (y^2 - 2ay\sqrt{1 + p^2})\, dx$$

can be made a maximum, provided the ordinate at an extremity is not permitted to be less than a certain positive value.

CHAPTER V.

PROBLEMS.

148. PROB. I. It is required to find what function y is of x, when

$$u \equiv \int_{x_0}^{x_1} (ax - y^3)\, y dx$$

is a maximum or a minimum.

Here $$N \equiv ax - 3y^2.$$

So that, for a maximum or a minimum, the equation,

$$ax - 3y^2 = I(x),$$

must be satisfied. This requires that

$$y = \sqrt{\frac{ax}{3}} + I',$$

or $$y = -\sqrt{\frac{ax}{3}} + I''.$$

It will be recognized that x_1, x_0, must both be positive, and that the positive sign gives a maximum, and the negative sign a minimum. In fig. 38 the surface, $z = (ax - y^2)\, y$, is traced for z positive; a change in the sign of y will change the sign of z: the curves numbered 0, 1, 2, 3, 4, are projections on the plane of xy of the intersections of the surface with planes parallel to that plane for which z has values 0, z_1, z_2, ... increasing in order.

In figs. 39, 40 are drawn the traces in the planes of xy and zx of a curve which makes $\int z dx$ a maximum. It will be readily seen that $\int z dx$ is the same for all such curves.

149. **Prob. II.** Having given

$$u \equiv \int_{x_0}^{x_1} (ax - y^2)\, y\, dx,$$

$$c = \int_{x_0}^{x_1} y\, dx,$$

it is required to find what function y is of x when u is a maximum or a minimum, c being a constant (126). Making v a maximum, where, λ being an arbitrary constant,

$$v \equiv \int_{x_0}^{x_1} (axy - y^3 + \lambda y)\, dx,$$

we have $$N \equiv ax - 3y^2 + \lambda;$$

$$\therefore y = \pm \sqrt{\frac{ax + \lambda + I(x)}{3}} = \pm \sqrt{\frac{ax + \lambda}{3}} + I'.$$

The + sign will be found to give the maximum, the − the minimum. λ is to be thus determined (we take + sign),

$$c = \int_{x_0}^{x_1} \sqrt{\frac{ax + \lambda}{3}}\, dx = \frac{2}{a}\left(\frac{ax_1 + \lambda}{3}\right)^{\frac{3}{2}} - \frac{2}{a}\left(\frac{ax_0 + \lambda}{3}\right)^{\frac{3}{2}}.$$

This will lead to a biquadratic in λ, as is seen by examining the equation

$$\left(\frac{ax_1 + \lambda}{3}\right)^3 + \left(\frac{ax_0 + \lambda}{3}\right)^3 - \frac{a^2c^2}{4} = 2\left(\frac{ax_1 + \lambda}{3}\right)^{\frac{3}{2}}\left(\frac{ax_0 + \lambda}{3}\right)^{\frac{3}{2}}.$$

Now, when ac is positive, the equations,

$$\left(\frac{ax_1 + \lambda}{3}\right)^{\frac{3}{2}} - \left(\frac{ax_0 + \lambda}{3}\right)^{\frac{3}{2}} = \frac{ac}{2}, \quad \left(\frac{ax_1 + \lambda}{3}\right)^{\frac{3}{2}} + \left(\frac{ax_0 + \lambda}{3}\right)^{\frac{3}{2}} = \frac{ac}{2},$$

have each only one real root, and

$$-\left(\frac{ax_1 + \lambda}{3}\right)^{\frac{3}{2}} \pm \left(\frac{ax_0 + \lambda}{3}\right)^{\frac{3}{2}} = \frac{ac}{2}$$

has no real root. So the biquadratic in λ has two real roots only.

The root of the first of these last three equations is the value which λ must have.

If $\int_{x_0}^{x_1} y\,dx = \int_{x_0}^{x_1} \sqrt{\frac{ax}{3}}\,dx$, $\lambda = 0$, and the solution is contained in Art. 148.

150. Prob. III. Having given

$$u \equiv \int_{x_0}^{x_1} x\,(x-y)^2\,dx,$$

to find y when u is a maximum or a minimum.

$$N \equiv -2x\,(x-y).$$

So that, for a maximum or a minimum, we must have

$$y = x + I(x).$$

If x_0, x_1, are both positive, we have a minimum: if both negative, a maximum; if different in sign, neither. The trace of the surface, $z = x\,(x-y)^2$, makes this clear (fig. 41).

151. Prob. IV. To find when

$$u \equiv \int_{x_0}^{x_1} \{2y + 3\,(x-y)^{\frac{2}{3}}\}\,dx$$

is a maximum or a minimum.

$$N \equiv 2 - 2\,(x-y)^{-\frac{1}{3}}; \quad \frac{dN}{dy} \equiv -\frac{2}{3}\,(x-y)^{-\frac{4}{3}}.$$

N is truly discontinuous when $x = y$, and then

$$\delta u = \int_{x_0}^{x_1} (2\delta y + 3\delta y^{\frac{2}{3}})\,dx = \int_{x_0}^{x_1} 3\delta y^{\frac{2}{3}}\,dx$$

initially, and is always positive. So

$$y = x + I(x) \text{ makes } u \text{ a minimum.}$$

Again, if $1-(x-y)^{-\frac{1}{2}}=I(x)$, or

$$y=x-1+I_1(x),$$

u is a maximum, for $\dfrac{dN}{dy}$ is negative (fig. 42). We trace all these surfaces by representing the projections of sections of the surface throughout each of which z is constant. If each of these projections be cut out in cardboard, of moderate thickness, and the cards be put one over the other properly in order, a very correct idea of the surface might be shaped.

152. Prob. V. To find when

$$u\equiv\int_{x_0}^{x_1}\{3y+2(x-y)^{\frac{3}{2}}\}\,dx$$

is a maximum or a minimum.

$$N\equiv 3-3(x-y)^{\frac{1}{2}},\quad \frac{dN}{dy}\equiv\frac{3}{2}(x-y)^{-\frac{1}{2}}.$$

When $\qquad x-y=1+I(x)$

(I must never be <-1), u is a minimum.

When $\qquad y=x+I,$

where I cannot be positive, N is truly discontinuous, for an increase in the value of y makes N imaginary.

In this case,

$$\delta u=\int_{x_0}^{x_1}\{3\delta y+2(-\delta y)^{\frac{3}{2}}\}\,dx,$$

δy must be negative, and so u is a maximum (fig. 43).

153. Prob. VI. To find when

$$u\equiv\int_{x_0}^{x_1}(y^4-4ay^3-4x^3y)\,dx$$

is a maximum or a minimum.

$$N\equiv 4(y^3-3ay^2-x^3);\quad \frac{dN}{dy}\equiv 12(y^2-2ay).$$

In fig. 44 the surface, $z = y^4 - 4ay^3 - 4x^3y$, is traced. In 110 it is shown that the roots of $N = 0$, as given by Tartaglia's rule, are truly discontinuous. But it is evident that $y =$ ordinate of OQ, or of RP ($ER = 2a$), makes u a minimum, and $y =$ ordinate of RO makes u a maximum (fig. 19). So that (Arts. 110, 79),

$$\left.\begin{aligned} \pi y &= \Big|^{\alpha=0} \left(y_1 \int_{x+h}^{+\infty} + y_3 \int_{-\infty}^{x+h}\right) \frac{\sin\alpha\, dz}{e^z - 2\cos\alpha + e^{-z}} + I_2(x), \\ \pi y &= \Big|^{\alpha=0} \left(y_2 \int_{x+k}^{+\infty} + y_1 \int_{-\infty}^{x+k}\right) \frac{\sin\alpha\, dz}{e^z - 2\cos\alpha + e^{-z}} + I_3(x), \end{aligned}\right\} \ldots\ldots (1),$$

render u a minimum, and

$$\pi y = \Big|^{\alpha=0} \left(y_3 \int_{x+h}^{+\infty} + y_1 \int_{x+k}^{x+h} + y_2 \int_{-\infty}^{x+k}\right) \frac{\sin\alpha\, dz}{e^z - 2\cos\alpha + e^{-z}} + I_1(x)$$

renders u a maximum, $-h$, $-k$ being the abscissæ for S', S respectively. Observe that x_0, x_1, must be respectively both negative, both intermediate to $-2^{\frac{2}{3}}a$ and $+\infty$, and both intermediate to $-2^{\frac{2}{3}}a$ and zero in the three cases.

Further, observe that there are points on each of the curves (1) for which $(y^4 - 4ay^3 - 4x^3y)$ has the same value for the same value of x, and so a truly discontinuous function can be readily assigned for y such that z be not truly discontinuous, and $\int_{x_0}^{x_1} z\,dx$ a minimum, x_0 and x_1 being entirely unrestricted. Just as the sky-line in a hill country can be continuous, although the outline on the hills which actually corresponds to it is truly discontinuous.

154. PROB. VII. Having given

$$u \equiv \int_{x_0}^{x_1} (\sqrt{1+p^2} + \lambda p)\, dx,$$

to find when u is a minimum.

$$P \equiv \frac{p}{\sqrt{1+p^2}} + \lambda.$$

λ^2 must be less than unity, and then

$$p = \frac{-\lambda}{\sqrt{1-\lambda^2}} + I_\infty(x)$$

will render u a minimum.

The curve $P_0Q_0Q_1P_1$ (fig. 45) will not render u a minimum, for, without altering $\int \lambda p dx$, $\int\sqrt{1+p^2}\,dx$ can be rendered smaller by taking δp negative for P_0Q_0 and positive for Q_1P_1, so as not to move P_0, P_1, nor change the length of Q_0Q_1. This holds whenever P_0Q_0, P_1Q_1 are not parallel. But in fig. 46, $P_0Q_0Q_1P_1$ renders u a minimum, if P_0Q_0, Q_1P_1 have each the proper inclination to Ox. For no variation, finite throughout, can alter the length of Q_0Q_1. So all that the Calculus of Variations can tell us concerning the shortest line joining two points, when the question is discussed by means of the above definite integral, is that, when the line joining them is not perpendicular to Ox, it has a constant inclination to Ox. It requires a further application of the methods of the Differential Calculus, or something equivalent, to show that, among all such lines, the line never perpendicular to Ox is the shortest.

155. PROB. VIII. A material point is so constrained in its motion that its velocity always $= u + \frac{pv}{\sqrt{1+p^2}}$, where $u > v$; it is required to find its path from one given point to another, when the time the journey takes is a minimum.

We have $$\frac{dx}{dt} = \frac{u\sqrt{1+p^2}+pv}{1+p^2}; \quad \frac{dy}{dx} = p;$$

so that $$t = \int_{x_0}^{x_1} \frac{(1+p^2)\,dx}{u\sqrt{1+p^2}+pv}; \quad y_1 - y_0 = \int_{x_0}^{x_1} p dx;$$

and we enquire if U is a minimum, where, for some value of λ, independent of x, but not yet known,

$$U \equiv \int_{x_0}^{x_1} \left\{ \frac{1+p^2}{u\sqrt{1+p^2}+pv} + \lambda p \right\} dx.$$

From this we find

$$P \equiv \frac{2p}{u\sqrt{1+p^2}+pv} - \frac{pu\sqrt{1+p^2}+(1+p^2)\,v}{(u\sqrt{1+p^2}+pv)^2} + \lambda.$$

Now the equation, $P=0$, gives,

$$-\frac{d\lambda}{dp} = \frac{3puv + (u^2+2v^2)\sqrt{1+p^2}}{\sqrt{1+p^2}\,(u\sqrt{1+p^2}+pv)^3}.$$

It is requisite for a minimum that this should throughout be positive. In figs. 47, 48, we trace the curves which are, when $P=0$, the locus of λ and p, when

$$u^2 - 3uv + 2v^2 \equiv (u-v)(u-2v)$$

is positive, and when it is negative; that is, according as $u >$ or $< 2v$.

In these curves p is always given without ambiguity, when λ and the sign of $\frac{d\lambda}{dp}$ are given. The portion of fig. 48, for which $\frac{d\lambda}{dp}$ is positive, requiring that $\frac{dP}{dp}$ be negative, does not correspond to a minimum solution. If the equation, $P=0$, be solved, it gives a quadratic in p^2, *i.e.* a biquadratic in p: the roots of which which are also roots of the equation

$$-pu\sqrt{1+p^2} + (p^2-1)\,v + \lambda\,(-u\sqrt{1+p^2}+pv)^2 = 0,$$

have nothing to do with the problem considered.

So, p being that root of $P=0$ for which $\frac{d\lambda}{dp}$ is $-$,

$$y - y_0 = p\,(x-x_0) + I(x)$$

is the equation to a minimum course. Among such the minima minimorum courses can be determined by Differential Calculus methods. They will require $I(x)=0$, unless $2v > u$ and

$$-\frac{y_1-y_0}{x_1-x_0} > \frac{u^2-uv+v^2}{\sqrt{u\,(u^2-v^2)\,(2v-u)}}.$$

In verification of the foregoing and proof of the last statement, we subjoin a geometrical illustration.

156. Let x_0, y_0, be the co-ordinates of P_0, x_1, y_1, of P_1. Then $p=\frac{y_1-y_0}{x_1-x_0}$, and, for course P_0P_1, where p is positive (fig. 49),

$$t=\int_{x_0}^{x_1}\frac{(1+p^2)\,dx}{u\sqrt{1+p^2}+pv}=\frac{(1+p^2)(x_1-x_0)}{u\sqrt{1+p^2}+pv}.$$

But, for course $P_0Q_0Q_1P_1$, we shall have,

$$t=\frac{P_0Q_0}{u-v}+\frac{P_1Q_1}{u-v}+\int_{x_0}^{x_1}\frac{\{1+(p+\delta p)^2\}\,dx}{u\sqrt{1+(p+\delta p)^2}+(p+\delta p)\,v},$$

and the difference between these is ultimately, when δp is positive,

$$(x_1-x_0)\left\{\frac{1}{u-v}+\frac{pu\sqrt{1+p^2}+(p^2-1)\,v}{(u\sqrt{1+p^2}+pv)^2}\right\}\delta p,$$

which is positive; when δp is negative, and Q_0Q_1 lie on opposite sides of P_0P_1, we have for the difference,

$$(x_1-x_0)\left\{\frac{1}{u+v}-\frac{pu\sqrt{1+p^2}+(p^2-1)\,v}{(u\sqrt{1+p^2}+pv)^2}\right\}(-\delta p),$$

which is always positive, for $\frac{1}{u+v}-\frac{p}{u\sqrt{1+p^2}+pv}$ is positive.

But if p be negative (fig. 50), for course $P_0Q_0Q_1P_1$ we shall have, δp positive,

$$t=\frac{(x_1-x_0)\,\delta p}{u-v}+\int_{x_0}^{x_1}\frac{(1+p^2)\,dx}{u\sqrt{1+p^2}+pv}+\int_{x_0}^{x_1}\frac{pu\sqrt{1+p^2}+(p^2-1)\,v}{(u\sqrt{1+p^2}+pv)^2}\,\delta p\,dx,$$

which will be a shorter course than the course P_0P_1, provided p be such that a negative value is given to

$$\frac{1}{u-v}+\frac{pu\sqrt{1+p^2}+(p^2-1)\,v}{(u\sqrt{1+p^2}+pv)^2}$$
$$\equiv\frac{(1+p^2)\,u^2+(p^2-1)\,uv+v^2+puv\sqrt{1+p^2}+pu^2\sqrt{1+p^2}}{(u-v)(u\sqrt{1+p^2}+pv)^2}.$$

Now, if we solve the equation,

$$u^2 - uv + v^2 + u(u+v)(p^2 + p\sqrt{1+p^2}) = 0,$$

we obtain
$$p = -\frac{u^2 - uv + v^2}{\sqrt{u(u^2 - v^2)(2v-u)}},$$

and
$$\frac{(u^2 - uv + v^2)^2}{u(u^2 - v^2)(2v-u)} < \frac{(u^2 + 2v^2)^2}{(u^2 - v^2)(4v^2 - u^2)},$$

if $(2v + u)(u^4 - 2u^3v + 3u^2v^2 - 2uv^3 + v^4) < u(u^4 + 4u^2v^2 + 4v^4)$,

i. e. $-u^3v^2 + 4u^2v^3 - 3uv^4 + 2v^5 < 4u^3v^2 + 4uv^4$,

which is true because

$$5u^3 - 4u^2v + 7uv^2 - 2v^3 \text{ is positive.}$$

It follows from this that for values of p between

$$-\frac{u^2 - uv + v^2}{\sqrt{u(u^2 - v^2)(2v-u)}}, \text{ and } -\frac{u^2 + 2v^2}{\sqrt{(u^2 - v^2)(4v^2 - u^2)}},$$

we never have a minimum minimorum course. If $\frac{y_1 - y_0}{x_1 - x_0}$ have such value, the minimum minimorum course is given, when simplest, by

$$y = -\frac{u^2 - uv + v^2}{\sqrt{u(u^2 - v^2)(2v-u)}}(x - x_0) + I + \eta.$$

I must be positive when $x = x_0$ and negative when $x = x_1$, and is determined so that $y = y_0$ when $x = x_0$ and $y = y_1$ when $x = x_1$. A slight examination will show that every value of I which satisfies these conditions gives the same value to the time of passage.

To illustrate the fact that there is not even a minimum when p has a value so that

$$-p > \frac{u^2 + 2v^2}{\sqrt{(u^2 - v^2)(4v^2 - u^2)}},$$

the line P_0P_1 may be conceived moved into the position of any locus through P_0P_1 not requiring δp to be initially infinite: $\delta^2 U$

will be found to be negative, and so the course not even a minimum.

The question still remains whether the method by which we have attacked this question, wherein we essentially restrict ourselves to those courses in which the distance from the axis of y never decreases, does not, owing to that restriction, prevent us from arriving at the true solution of the physical question with which we started. It will naturally be expected that this is the case, when, in the equation to the minimum minimorum course, I has value. As, however, this question is completely met in the problem which we will next consider, we will not discuss it further here.

157. Prob. IX. A material point is so constrained in its motion that its velocity always $= u + \dfrac{pv \cos\alpha + v \sin\alpha}{\sqrt{1+p^2}}$, where $u > v$; it is required to find its path from one given point on the axis of x to another when the time the journey takes is a minimum.

$$\text{We have } 0 = \int_{x_0}^{x_1} p\,dx\,;\quad t = \int_{x_0}^{x_1} \frac{(1+p^2)\,dx}{u\sqrt{1+p^2} + pv\cos\alpha + v\sin\alpha}\,;$$

and we must have, for some value of λ, independent of x, but not yet known,

$$U = \text{minimum} = \int_{x_0}^{x_1} \left\{ \frac{1+p^2}{u\sqrt{1+p^2} + pv\cos\alpha + v\sin\alpha} + \lambda p \right\} dx.$$

$$\text{Now } P \equiv \frac{pu\sqrt{1+p^2} + (p^2-1)\,v\cos\alpha + 2pv\sin\alpha}{(u\sqrt{1+p^2} + pv\cos\alpha + v\sin\alpha)^2} + \lambda,$$

$$\frac{dP}{dp} = \frac{u(1+p^2) + up^2 + 2(pv\cos\alpha + v\sin\alpha)\sqrt{1+p^2}}{\sqrt{1+p^2}\,(u\sqrt{1+p^2} + pv\cos\alpha + v\sin\alpha)^2}$$

$$- \frac{2\{pu\sqrt{1+p^2} + (p^2-1)\,v\cos\alpha + 2pv\sin\alpha\}\{pu + v\cos\alpha\sqrt{1+p^2}\}}{\sqrt{1+p^2}\,(u\sqrt{1+p^2} + pv\cos\alpha + v\sin\alpha)^3}$$

$$=\frac{(pv\cos\alpha+v\sin\alpha)(3+4p^2)u+\{u^2(1+2p^2)+2(pv\cos\alpha+v\sin\alpha)^2\}\sqrt{1+p^2}}{\sqrt{1+p^2}\,(u\sqrt{1+p^2}+pv\cos\alpha+v\sin\alpha)^3}$$

$$-\frac{4p^3uv\cos\alpha+4p^2uv\sin\alpha+2\{p^2u^2+(p^2-1)v^2\cos^2\alpha+2pv^2\sin\alpha\cos\alpha\}\sqrt{1+p^2}}{\sqrt{1+p^2}\,(u\sqrt{1+p^2}+pv\cos\alpha+v\sin\alpha)^3}$$

$$=\frac{3(pv\cos\alpha+v\sin\alpha)\,u+(u^2+2v^2\cos^2\alpha+2v^2\sin^2\alpha)\sqrt{1+p^2}}{\sqrt{1+p^2}\,(u\sqrt{1+p^2}+pv\cos\alpha+v\sin\alpha)^3}$$

$$=\frac{3(p\cos\alpha+\sin\alpha)\,uv+(u^2+2v^2)\sqrt{1+p^2}}{\sqrt{1+p^2}\,(u\sqrt{1+p^2}+pv\cos\alpha+v\sin\alpha)^3}.$$

It is requisite, for a minimum, that this should be always positive. The numerator will vanish, if

$$(u^2+2v^2)\sqrt{(p\cos\alpha+\sin\alpha)^2+(p\sin\alpha-\cos\alpha)^2}$$
$$+3(p\cos\alpha+\sin\alpha)\,uv=0,$$

or, when $$\left(\frac{p\sin\alpha-\cos\alpha}{p\cos\alpha+\sin\alpha}\right)^2=\frac{9u^2v^2}{(u^2+2v^2)^2}-1,$$

i. e. $$\left(\tan\alpha-\frac{1}{\cos\alpha\,(p\cos\alpha+\sin\alpha)}\right)^2=\frac{(u^2-v^2)(4v^2-u^2)}{(u^2+2v^2)^2},$$

$$\frac{1}{p\cos\alpha+\sin\alpha}=\sin\alpha\pm\frac{\cos\alpha\sqrt{(u^2-v^2)(4v^2-u^2)}}{(u^2+2v^2)}$$

is negative. Now there will be two such values for p when $\sin\alpha$ is negative, and

$$\tan^2\alpha>\frac{(u^2-v^2)(4v^2-u^2)}{(u^2+2v^2)^2},$$

i.e. $$\cos^2\alpha<\frac{(u^2+2v^2)^2}{9u^2v^2}.$$

Again, when $\lambda = \dfrac{\mp 1}{u \pm v \cos\alpha}$, we have, if $P = 0$,

$$
\begin{aligned}
pu\,(u \pm v\cos\alpha)\sqrt{1+p^2} \mp 2uv\,(p\cos\alpha + \sin\alpha)\sqrt{1+p^2}& \\
= \pm u^2(1+p^2) \pm v^2(p\cos\alpha+\sin\alpha)^2& \\
-(u \pm v\cos\alpha)\left\{(p^2-1)\,v\cos\alpha + 2pv\sin\alpha\right\};&
\end{aligned}
$$

or,
$$
\begin{aligned}
(pu^2 \mp puv\cos\alpha \mp 2uv\sin\alpha)\sqrt{1+p^2}& \\
= \pm(u^2 \mp uv\cos\alpha)\,p^2 - 2puv\sin\alpha& \\
\pm(u^2 \pm uv\cos\alpha + v^2),&
\end{aligned}
$$

giving
$$
\begin{aligned}
p^2\{(u^2 \mp uv\cos\alpha)^2 + 4u^2v^2\sin^2\alpha& \\
-2\,(u^2 \mp uv\cos\alpha)(u^2 \pm uv\cos\alpha + v^2) - 4u^2v^2\sin^2\alpha\}& \\
+2p\,\{2u^2v^2\sin\alpha\cos\alpha \mp 2u^3v\sin\alpha \pm 2uv\sin\alpha\,(u^2 \pm uv\cos\alpha + v^2)\}& \\
+4u^2v^2\sin^2\alpha - (u^2 \pm uv\cos\alpha + v^2)^2 = 0,&
\end{aligned}
$$

or,
$$
\begin{aligned}
u\,(u \mp v\cos\alpha)(u^2 \pm 3uv\cos\alpha + 2v^2)\,p^2& \\
-4uv^2\sin\alpha\,(2u\cos\alpha \pm v)\,p& \\
+(u^2 \pm uv\cos\alpha + v^2)^2 - 4u^2v^2\sin^2\alpha = 0.&
\end{aligned}
$$

Taking the upper sign, as it will be observed that changing the sign of v will change from the upper sign to the lower sign, I find

$$
\begin{aligned}
\{u\,(u - v\cos\alpha)(u^2 + 3uv + 2v^2)\,p - 2uv^2\sin\alpha\,(2u\cos\alpha + v)\}^2& \\
= 4u^2v^4\sin^2\alpha\,(4u^2\cos^2\alpha + 4uv\cos\alpha + v^2)& \\
-u\,(u^3 + 2u^2v\cos\alpha + 2uv^2 - 3uv^2\cos^2\alpha - 2v^3\cos\alpha).& \\
(u^4 + 2u^3v\cos\alpha + 3u^2v^2\cos^2\alpha - 2u^2v^2\sin^2\alpha + 2uv^3\cos\alpha + v^4)& \\
= -u\,\{u^7 + 4u^6v\cos\alpha + (2 - 3\cos^2\alpha + 4\cos^2\alpha + 3\cos^2\alpha - 2\sin^2\alpha)\,u^5v^2& \\
+(-2\cos\alpha + 4\cos\alpha - 6\cos^3\alpha + 6\cos^3\alpha - 4\cos\alpha\sin^2\alpha + 2\cos\alpha)\,u^4v^3&
\end{aligned}
$$

$$\begin{aligned}
&+ (-16 \sin^2\alpha \cos^2\alpha + 1 + 4 \cos^2\alpha + 6 \cos^4\alpha - 4 \sin^2\alpha - 9 \cos^4\alpha \\
&\qquad\qquad + 6 \sin^2\alpha \cos^2\alpha - 4 \cos^2\alpha)\, u^3v^4 \\
&+ (-16 \sin^2\alpha \cos\alpha + 2 \cos\alpha + 4 \cos\alpha - 6 \cos^3\alpha - 6 \cos^3\alpha \\
&\qquad\qquad + 4 \sin^2\alpha \cos\alpha)\, u^2v^5 \\
&+ (-4 \sin^2\alpha + 2 - 3 \cos^2\alpha - 4 \cos^2\alpha)\, uv^6 - 2v^7 \cos\alpha\} \\
&= -u\, (u^7 + 4u^6v \cos\alpha + 6u^5v^2 \cos^2\alpha + 4u^4v^3 \cos^3\alpha \\
&\qquad - 2u^3v^4 \cos^4\alpha - 6u^3v^4 \sin^2\alpha \cos^2\alpha \\
&\qquad - 3u^3v^4 \sin^4\alpha - 6u^2v^5 \sin^2\alpha \cos\alpha \\
&\qquad - 6u^2v^5 \cos^3\alpha - 2uv^6 \sin^2\alpha - 5uv^6 \cos^2\alpha - 2v^7 \cos\alpha) \\
&= -u\, \{u^3 (u + v \cos\alpha)^4 - 3u^3v^4 - 2v^6 (u + v \cos\alpha) \\
&\qquad\qquad - 3uv^6 \cos^2\alpha - 6u^2v^5 \cos\alpha\} \\
&= -u\, (u + v \cos\alpha)\, \{u^3 (u + v \cos\alpha)^3 - 3u\, (u + v \cos\alpha)\, v^4 - 2v^6\} \\
&= -u\, (u + v \cos\alpha)\, (u^2 + uv \cos\alpha - 2v^2)\, (u^2 + uv \cos\alpha + v^2)^2.
\end{aligned}$$

By aid of the foregoing investigations we can trace the curve which is the locus of λ and p when $P = 0$.

If $u > 2v$, $\dfrac{d\lambda}{dp}$ does not vanish for finite values of p, and the curve will resemble fig. (51).

If $u < 2v$, $\sin\alpha +$, $9u^2v^2 \cos^2\alpha < (u^2 + 2v^2)^2$, the curve will also resemble fig. (51).

If $u < 2v$, $\sin\alpha +$, $9u^2v^2 \cos^2\alpha > (u^2 + 2v^2)^2$, then since

$$(u^2 + 2v^2) > 3\, (u^2 - 2v^2), \text{ or } > 3\, (2v^2 - u^2),$$

$$(u^2 - uv \cos\alpha - 2v^2) \text{ and } (u^2 + uv \cos\alpha - 2v^2)$$

have opposite signs, and the curve will resemble fig. (54), when $(u^2 - uv \cos\alpha - 2v^2)$ is negative, and the curve can similarly be traced when $(u^2 + uv \cos\alpha - 2v^2)$ is negative, when it will cut the other asymptote.

If $u < 2v$, $\sin\alpha -$, $(u^2 - uv \cos\alpha - 2v^2)$, $(u^2 + uv \cos\alpha - 2v^2)$ both positive, so that $9u^2v^2 \cos^2\alpha < (u^2 + 2v^2)^2$, the curve will resemble fig. (55).

If $u < 2v$, $\sin\alpha -$, $(u^2 - uv\cos\alpha - 2v^2)$, $(u^2 + uv\cos\alpha - 2v^2)$ both negative, the curve will resemble fig. (56).

But if $u < 2v$, $\sin\alpha -$,

$$(u^2 - uv\cos\alpha - 2v^2), \text{ and } (u^2 + uv\cos\alpha - 2v^2)$$

different in sign, the curve will cut only one asymptote, but will cut it in two places, resembling figs. (52) or (53).

A comparison of these several cases shows that sometimes p has, for the same value of λ, two different values for both of which $\frac{d\lambda}{dp}$ is negative, and that this is always the case when $\Big|^{p=0} \frac{d\lambda}{dp}$ is positive.

This shows the existence, then, of an infinite number of routes, along none of which p is ever infinite, and for all of which the time of journey is a minimum. It will be observed that for each individual route $(t + \lambda\int_{x_0}^{x_1} p\,dx)$ is rendered a minimum independent of the condition of $y = y_1$ when $x = x_1$; but $(t + \lambda'\int_{x_0}^{x_1} p\,dx)$, where $\lambda' \neq \lambda$, is not rendered a minimum independent of that condition. The method of the Calculus of Variations is, as such, unable to compare the time of journey along one of these routes with the time of journey along another. For, if a curve move (fig. 57) out of the position $X_0P_1X_1$ into $X_0P_2X_1$, where $X_0P_1X_1$, $X_0P_2X_1$ are triangles, and P_1P_2 not ultimately perpendicular to X_0X_1, δp is not infinitely small throughout in the limit when P_1P_2 vanishes. And so the method of Art. 123 (and every method based, avowedly or unavowedly, on expansion by Taylor's Theorem), in which higher powers of δp are neglected in comparison with lower powers, cannot recognize a route changing its form from $X_0P_1X_1$ to a form indefinitely near, $X_0P_2X_1$, without passing through an intermediate form for which the time of journey is greater than for either $X_0P_1X_1$, $X_0P_2X_1$, notwithstanding that the difference between these times may be made as small as we please by taking P_1P_2 small enough. It is for this reason that the words 'all such forms...as are initially evanescent,' are introduced in the fifth and sixth lines of Art. 123. The Differential Calculus, in like manner, would refuse to recognize every point in the line, $y = \Big|^{a=\infty} \frac{\sin ax}{a}$, to be at a maximum or minimum distance from the axis of x, as every point in the line, $y = 0$, is; and would only concede that distinction to the points whose abscissæ are odd multiples of $\frac{\pi}{2a}$.

In like manner as the Differential Calculus does not, as such, tell us which is the absolutely greatest and least among those maxima and minima which its methods discover; leaving that question to be decided *per enumerationem simplicem:* so the Calculus of Variations does not, as such, determine which are the greatest and least among those maxima and minima which its method discovers, leaving that question to be decided by the methods of the Differential Calculus, and *per enumerationem simplicem.*

Thus, in the present Problem, it must not be supposed that the time of journey along each of the routes, for which that time has been shown to be a minimum, is the same. The methods of the Differential Calculus must be employed if we wish to find the minima minimorum among these minima.

158. In order to find the minimum minimorum course we must apply Differential Calculus methods. For this purpose, suppose (fig. 58) the course of the material point to be the path X_0PX_1. Then

$$\text{velocity along } X_0P = u + v\sin(\alpha+\theta),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ PX_1 = u + v\sin(\alpha-\phi), \text{ and}$$

$$t = \frac{X_0P}{u+v\sin(\alpha+\theta)} + \frac{PX_1}{u+v\sin(\alpha-\phi)}$$

$$= \frac{X_0X_1}{\sin(\theta+\phi)}\left\{\frac{\sin\phi}{u+v\sin(\alpha+\theta)} + \frac{\sin\theta}{u+v\sin(\alpha-\phi)}\right\}.$$

This must be made a minimum by the variation of θ and ϕ.

I find

$$0=\frac{-\sin\phi\cos(\theta+\phi)\,\{u+v\sin(\alpha+\theta)\}-v\sin(\theta+\phi)\sin\phi\cos(\alpha+\theta)}{\sin^2(\theta+\phi)\,\{u+v\sin(\alpha+\theta)\}^2}$$

$$+\frac{\cos\theta\sin(\theta+\phi)-\sin\theta\cos(\theta+\phi)}{\sin^2(\theta+\phi)\,\{u+v\sin(\alpha-\phi)\}};$$

whence, $\dfrac{u+v\sin(\alpha+\theta)}{u+v\sin(\alpha-\phi)}=\dfrac{u\cos(\theta+\phi)+v\sin(\alpha+2\theta+\phi)}{u+v\sin(\alpha+\theta)}.$

In like manner

$$\frac{u+v\sin(\alpha-\phi)}{u+v\sin(\alpha+\theta)}=\frac{u\cos(\theta+\phi)+v\sin(\alpha-\theta-2\phi)}{u+v\sin(\alpha-\phi)}.$$

So that

$$\frac{u+v\sin(\alpha+\theta)}{u+v\sin(\alpha-\phi)}=\frac{u\{1+\cos(\theta+\phi)\}+2v\sin(\alpha+\theta)\cos(\theta+\phi)}{u\{1+\cos(\theta+\phi)\}+2v\sin(\alpha-\phi)\cos(\theta+\phi)}$$

$$=\frac{v\sin(\alpha+\theta)-v\sin(\alpha+\theta)\cos(\theta+\phi)}{v\sin(\alpha-\phi)-v\sin(\alpha-\phi)\cos(\theta+\phi)}=1.$$

So that $\quad \alpha+\theta=\pi-\alpha+\phi=\dfrac{\pi+\theta+\phi}{2}$, and

$$u+v\sin(\alpha-\phi)=-u\cos 2(\alpha-\phi)-v\sin 3(\alpha-\phi),$$

$$2u\cos^2(\alpha-\phi)+2v\sin 2(\alpha-\phi)\cos(\alpha-\phi)=0,$$

if $\cos(\alpha-\phi)=0$, or $u+2v\sin(\alpha-\phi)=0$.

This last solution lies among the solutions given by the method of the Calculus of Variations, for, from our equations,

$$\left\{\frac{u+v\sin(\alpha+\theta)}{u+v\sin(\alpha-\phi)}\right\}^2$$

$$=\frac{-\sin\phi\,\{u\sin\theta-v\cos(\alpha+2\theta)\}+\cos\phi\,\{u\cos\theta+v\sin(\alpha+2\theta)\}}{\cos\phi\,\{u\cos\phi+v\sin(\alpha-2\phi)\}+\sin\phi\,\{u\sin\phi+v\cos(\alpha-2\phi)\}}$$

$$=\frac{\sin\theta\,\{u\sin\theta-v\cos(\alpha+2\theta)\}+\cos\theta\,\{u\cos\theta+v\sin(\alpha+2\theta)\}}{\cos\theta\,\{u\cos\phi+v\sin(\alpha-2\phi)\}-\sin\theta\,\{u\sin\phi+v\cos(\alpha-2\phi)\}}$$

$$=\frac{-\sin(\theta+\phi)\,\{u\sin\theta-v\cos(\alpha+2\theta)\}}{\sin(\theta+\phi)\,\{u\sin\phi+v\cos(\alpha-2\phi)\}}.$$

If on X_0X_1 be described an arc of a circle containing an angle equal to the value assigned by the above equations to $(\theta+\phi)$, and a line be drawn through the minimum ordinated point at right angles to the direction of fastest and slowest motion, its intersection with the circle will determine P.

As negative values for θ and ϕ are inadmissible, we see that, α ranging from $-\pi$ to $+\pi$, $(\alpha+\theta)$ must lie between $-\frac{\pi}{2}$ and 0, and $(\alpha-\phi)$ between $-\pi$ and $-\frac{\pi}{2}$. So $\sin\alpha$ must be negative. An investigation of second differential coefficients would lead to the same conclusions as we have already obtained from considering the locus of λ and p when $P=0$.

It will readily appear that, if a curve be drawn from X_0 to X_1, so that for it p^2 always less than in X_0PX_1, and any number of points be taken on it, two consecutive ones being made the opposite corners of a parallelogram whose sides are parallel to X_0P_1 PX_1, as in fig. 59, a route will be obtained having for its time of journey the same length as the route X_0PX_1 has, and the number of corners of parallelograms may be made as great as ever we please, and so the route be made to coincide with the curve throughout in position, although their tangents do not coincide.

It should be remarked that $\theta=\phi=0$ sometimes, but not always, renders t a minimum.

For $$\frac{1}{2\cos\theta}\left\{\frac{1}{u+v\sin(\alpha+\theta)}+\frac{1}{u+v\sin(\alpha-\theta)}\right\}$$

$$=\frac{1}{2\cos\theta}\left\{\frac{2u+2v\sin\alpha\cos\theta}{(u+v\sin\alpha\cos\theta)^2-v^2\cos^2\alpha\sin^2\theta}\right\}$$

$$=\frac{1}{1-\frac{1}{2}\theta^2}\cdot\frac{u+v\sin\alpha-\frac{1}{2}v\sin\alpha\,.\,\theta^2}{(u+v\sin\alpha-\frac{1}{2}v\sin\alpha\,.\,\theta^2)^2-v^2\cos^2\alpha\,.\,\theta^2}\text{ initially,}$$

$$=\frac{(1+\frac{1}{2}\theta^2)(u+v\sin\alpha-\frac{1}{2}v\sin\alpha\,.\,\theta^2)}{(u+v\sin\alpha)^2-(v^2+uv\sin\alpha)\,\theta^2}\text{ initially,}$$

$$= \frac{1}{u+v\sin\alpha}(1+\tfrac{1}{2}\theta^2)\left\{1-\frac{v\sin\alpha\,.\,\theta^2}{2(u+v\sin\alpha)}\right\}$$

$$\left\{1+\frac{v(v+u\sin\alpha)\,\theta^2}{(u+v\sin\alpha)^2}\right\} \text{ initially,}$$

and $(u+v\sin\alpha)^2-(u+v\sin\alpha)\,v\sin\alpha+2v\,(v+u\sin\alpha)$

$= u^2+3uv\sin\alpha+2v^2$, which is not always positive.

159. Prob. X. Having given

$$u \equiv \int_{x_0}^{x_1}(y+c\sqrt{1+p^2})\,dx,$$

$$y_1-y_0=\int_{x_0}^{x_1}p\,dx,$$

to find y when u is a minimum. We suppose c, y_1-y_0, to be positive finite constants, and the value of y to be y_0 when $x=x_0$, otherwise, by decreasing y without altering p, u may be made to decrease always. We have here to make (Art. 126) a minimum

$$v \equiv \int_{x_0}^{x_1}(y+\lambda p+c\sqrt{1+p^2})\,dx,$$

in which $\quad N\equiv 1,\ \ P\equiv\lambda+\dfrac{cp}{\sqrt{1+p^2}}$;

so that we must have

$$\left.\begin{aligned}\lambda+\frac{cp}{\sqrt{1+p^2}}-x+A=I(x),\\ \Big|_{x_0}^{x_1}\delta y\int N\,dx=0.\end{aligned}\right\}$$

The last equation is satisfied since $\Big|^{x=x_0}\delta y=0$, $\Big|^{x=x_1}\delta y=0$, for $y=y_0$, and $y=y_1$, for these values of x.

Now, $y_1 - y_0 = \int_{x_0}^{x_1} p\,dx = \int_{x_0}^{x_1} \left\{ \frac{x - A - \lambda + I}{\sqrt{c^2 - (x - A - \lambda + I)^2}} \right\} dx.$

So if we have always $c^2 > (x - A - \lambda + I)^2$, then

$$\int \left\{ \frac{x - A - \lambda + I}{\sqrt{c^2 - (x - A - \lambda + I)^2}} - \frac{x - A - \lambda}{\sqrt{c^2 - (x - A - \lambda)^2}} \right\} dx = 0,$$

and our solution will be

$$\left. \begin{aligned} y - y_0 &= -\sqrt{c^2 - (x - A - \lambda)^2} + \sqrt{c^2 - (x_0 - A - \lambda)^2}, \\ p &= \frac{x - A - \lambda + I}{\sqrt{c^2 - (x - A - \lambda + I)^2}}; \end{aligned} \right\}$$

$A + \lambda$ being determined from

$$y_1 - y_0 = -\sqrt{c^2 - (x_1 - A - \lambda)^2} + \sqrt{c^2 - (x_0 - A - \lambda)^2}.$$

Thus, if the circle through P and Q (fig. 60) with radius c, and centre on the side of PQ remote from X_0X_1, cut neither X_0P nor X_1Q on the side of PQ towards X_0X_1 ($OX_0 = x_0$, $OX_1 = x_1$, $X_0P = y_0$, $X_1Q = y_1$), then throughout the arc PRQ we have $c^2 > (x - A - \lambda)^2$, and the above investigation shows that u is, in this case, a minimum.

But if ever $\qquad c^2 = (x - A - \lambda + I)^2,$

we may have

$$\int_{x_0}^{x} \left\{ \frac{x - A - \lambda + I}{\sqrt{c^2 - (x - A - \lambda + I)^2}} - \frac{x - A - \lambda}{\sqrt{c^2 - (x - A - \lambda)^2}} \right\} dx = W(x),$$

where, Art. 136, $\dfrac{dW(x)}{dx} = I_\infty$.

Now, since $\qquad y_1 - y_0 = \int_{x_0}^{x_1} p\,dx,$

$$\begin{aligned} \therefore \quad W(x_1) &= y_1 - y_0 - \int_{x_0}^{x_1} \frac{x - A - \lambda}{\sqrt{c^2 - (x - A - \lambda)^2}}\,dx \\ &= y_1 - y_0 + \sqrt{c^2 - (x_1 - A - \lambda)^2} - \sqrt{c^2 - (x_0 - A - \lambda)^2}, \end{aligned}$$

the equation to determine $(A + \lambda)$.

In this case our solution is contained by the equations

$$\left.\begin{aligned} y-y_0 &= W(x)-\sqrt{c^2-(x-A-\lambda)^2}+\sqrt{c^2-(x_0-A-\lambda)^2},\\ \frac{dy}{dx} &= \frac{x-A-\lambda+I}{\sqrt{c^2-(x-A-\lambda+I)^2}}. \end{aligned}\right\}$$

It should be observed that no always finite value for δp can render

$$\int_{x_0}^{x_1} \delta p\, I(x)\, dx$$

other than zero. But if δp be allowed to be infinite, and δy be not, since, when δx is small enough,

$$\int_{x'}^{x'+\delta x} \delta p\, I(x)\, dx = \Big|^{x=x'} \delta p\,.\,\delta x\,.\,I(x'),$$

and
$$\int_{x'}^{x'+\delta x} \frac{d\delta y}{dx}\, dx = \Big|^{x=x'} \delta p \delta x = \Big|_{x'}^{x'+\delta x} \delta y,$$

we have
$$\int_{x'}^{x'+\delta x} \delta p\, I(x)\, dx = I(x') \Big|_{x'}^{x'+\delta x} \delta y,$$

so that
$$\int_{x_0}^{x_1} \delta p\, I(x)\, dx = \Sigma\, I(x') \Big|_{x'}^{x'+\delta x} \delta y,$$

with the condition
$$\int_{x_0}^{x_1} \delta p dx = \Big|_{x_0}^{x_1} \delta y = 0,$$

and $\int_{x_0}^{x_1} \delta p\, I(x)\, dx$ will not vanish unless $\Sigma\, I(x') \Big|_{x'}^{x'+\delta x} \delta y$ vanishes.

Thus if we have a curve $Pp_1p_2p_3Q$ (fig. 61), where Pp_1, p_2p_3 are arcs of circles with radius c, and centres having the same ordinate, and p_1p_2, Qq_3 are parallel to Oy, no finite change in δp can do anything but increase u for this curve, whose equations may be written as above, and

$$I = \lambda + A - \xi - c, \quad \text{when } x = \xi,$$

and
$$I = \lambda + A - x_1 + c, \quad \text{when } x = x_1,$$

and $I=0$, for values of x finitely distant from ξ, x_1. In this case we shall have, δp being taken to vanish when I vanishes, and ∞ when I does not vanish,

$$\delta v = \int_{x_0}^{x_1} \delta p \, I(x)\, dx = (\lambda + A - \xi - c)(-\delta y) + (\lambda + A + c - x_1)(\delta y)$$

$$= (2c + \xi - x_1)\, \delta y,$$

and in this case v can be diminished. But the Calculus of Variations cannot recognize such a variation of v.

It will be observed that when $W(x)$ changes its value upon $(x - A - \lambda) = c$, we have undoubtedly a minimum. But, when this occurs at any other point in the curve, u may be diminished, but it requires an infinite change in δp to produce an indefinitely small change in u.

Thus, if PQ' (fig. 62) be the arc of a circle, the ordinate of whose centre equals $X_1 Q'$, and whose radius is c, the curve $PQ'Q$ will undoubtedly be a minimum.

For this curve we have the equations

$$y - C_1 = W - \sqrt{c^2 - (x - A - \lambda)^2};$$

$$\frac{dy}{dx} = \frac{x - A - \lambda + I}{\sqrt{c^2 - (x - A - \lambda + I)^2}};$$

where

$$y_0 - C_1 = -\sqrt{c^2 - (x_0 - A - \lambda)^2};$$

$$x_1 - A - \lambda = c;$$

I must be zero when $x = x_1$, and must never elsewhere make $\frac{dy}{dx}$ so infinite as to affect y.

The value of W may be represented by

$$\Big|^{\alpha=0} \frac{2(y_1 - c_1)}{\pi} \int_{x-x_1}^{x} \frac{\sin\alpha \, . \, dz}{e^{z-x_1} - 2\cos\alpha + e^{-z+x_1}}.$$

By tracing the locus of the curve whose ordinate is $\frac{dy}{dx}$, or

$$\frac{x-A-\lambda+I}{\sqrt{c^2-(x-A-\lambda+I)^2}} \text{ (fig. 63)},$$

we see that the instantaneous values of I can have no effect either on $\int pdx$ or on $\int\sqrt{1+p^2}dx$.

160. **Prob. XI.** Among all surfaces of revolution of the same superficial area to find the ones whose solid content is a maximum.

If, in attacking this question, we refer the surface to rectangular co-ordinates, the axis of revolution being the axis of x, we shall have to investigate the maximum values of

$$u=\int_{x_0}^{x_1}\{y^2-2a\sqrt{y^2(1+p^2)}\}\,dx,$$

where a has a value, as yet unknown, independent of x.

Suppose consistent functions are found for y and p such that, for a particular value of a, u is rendered a maximum. Then, for all such variations as are consistent with $\int_{x_0}^{x_1}\sqrt{y^2(1+p^2)}\,dx$ being constant, these values for y and p will render $\int_{x_0}^{x_1}y^2dx$ a maximum. But the converse of this is not true. That is to say, consistent functions may be found, or exist, for y and p, such that, when $\int_{x_0}^{x_1}\sqrt{y^2(1+p)^2}\,dx$ is constant, $\int_{x_0}^{x_1}y^2dx$ is a maximum, although these functions do not render u a maximum. For we have shown (Art. 141), that when

$$p=\frac{-x}{\sqrt{4a^2-x^2}};\quad y=\sqrt{4a^2-x^2};$$

$\int_{-a}^{+a}\{y^2-2a\sqrt{y^2(1+p^2)}\}\,dx$ is not rendered a maximum. This we did by assigning to δy, δp, values which, it should be observed, increase $\int_{-a}^{+a}\sqrt{y^2(1+p^2)}\,dx$. Yet these values for y and p do render $\int_{-a}^{+a}y^2dx$ a maximum when $\int_{-a}^{+a}y^2\sqrt{1+p^2}\,dx$ is constant in value. For, Art. 141, y being positive,

$$\delta^2u=\int_{-a}^{+a}\left\{2\delta y^2-\frac{4ap}{\sqrt{1+p^2}}\,\delta y\delta p-\frac{2ay}{(1+p^2)^{\frac{3}{2}}}\,\delta p^2\right\}dx$$

$$=\int_{-a}^{+a}-\frac{(1+p^2)^{\frac{3}{2}}}{2a}\left\{\frac{2y^{\frac{1}{2}}\delta y}{p}-\frac{2a\delta y}{py^{\frac{1}{2}}\sqrt{1+p^2}}+\frac{2ay^{\frac{1}{2}}}{(1+p^2)^{\frac{3}{2}}}\delta p\right\}^2dx$$

$$+\int_{-a}^{+a}\frac{d}{dx}\left(\frac{2y-2a\sqrt{1+p^2}}{p}\,\delta y^2\right)dx$$

$$+\int_{-a}^{+a}\left\{2+\frac{2y\,(1+p^2)^{\frac{3}{2}}}{ap^2}-\frac{4(1+p^2)}{p^2}+\frac{2a\sqrt{1+p^2}}{p^2y}\right.$$

$$\left.+\left(\frac{2y-2a\sqrt{1+p^2}}{p^2}+\frac{2a}{\sqrt{1+p^2}}\right)\frac{dp}{dx}-2\right\}\delta y^2dx$$

$$+\int_{-a}^{+a}\left(-\frac{4ap}{\sqrt{1+p^2}}+\frac{4y}{p}-\frac{4a}{p\sqrt{1+p^2}}-\frac{4y-4a\sqrt{1+p^2}}{p}\right)\delta y\delta p.$$

In this the last integral vanishes identically, the second owing to the limits being fixed, and the third is

$$\int_{-a}^{+a}\left\{\frac{1+p^2}{p^2y}-\frac{(1+p^2)^{\frac{3}{2}}}{ap^2}\right\}\left\{\frac{2a}{\sqrt{1+p^2}}-2y-\frac{2ay}{(1+p^2)^{\frac{3}{2}}}\,\frac{dp}{dx}\right\}\delta y^2dx,$$

which vanishes for the circle in question [vide Art. 141, (6)]. So the first only remains, which is essentially negative, the part under the integral never becoming infinite, and it cannot vanish unless δy, δp have values such as will increase or decrease

$$\int_{-a}^{+a}\sqrt{y^2(1+p^2)}\,dx.$$

In like manner y equal any of the solutions of $\delta u = 0$ given in Art. 141 is a solution of this question. So (Art. 139) either $y = 0$, or y equal the ordinate of a falsely discontinuous curve, which in one part coincides with the axis of x, and in another part with a curve for which $\delta u = 0$, gives a solution. In every case a must be determined so that the surface of revolution have the given value. The number of solutions is infinite, variations in which δp is throughout finite alone being permitted. Our demonstration that the sphere solves the question, when the superficial area =, or $<$, $\pi (x_1 - x_0)^2$, equally applies, *mutatis mutandis*, to show that the curve corresponding to any of those which satisfy equation (4) of Art. 141, when $I(x) = 0$ throughout, solves the question, if the superficial area produced by its revolution has the right magnitude.

161. Observe that the definite integral, $\int (y^2 - 2ay\sqrt{1+p^2})\, dx$, does not exactly correspond to the question proposed. For the area of a surface of revolution is always $\int_{x_0}^{x_1} \sqrt{y^2(1+p^2)}\, dx$, and is only $\int_{x_0}^{x_1} y\sqrt{1+p^2}\, dx$ when y is positive. Also in this method of attacking the question the surface of revolution is restricted to lie between two planes, whose equations are $x = x_0$, $x = x_1$, and to be of such a nature that y has only one value for a given value of x.

162. But if in attacking this question we refer the surface to polar co-ordinates, the pole being in the axis of revolution, although we thereby confine our attention to such surfaces of revolution as can be seen entirely from some one point on the axis of revolution, to the exclusion of such surfaces as that generated by the revolution of a figure, for which r has more than one value when θ is given, yet such restriction is unimportant, as a little consideration will show that for no such excluded surface is the solid enclosed absolutely a maximum. We shall, accordingly, have to investigate the maximum values of

$$u = \int_0^\pi \left(r^3 \sin\theta - 3ar \sin\theta \sqrt{r^2 + \frac{dr^2}{d\theta}}\right) d\theta$$

For which, $\left(r' = \frac{dr}{d\theta}\right)$,

$$\delta u = \int_0^\pi \left\{ \left(3r^2 \sin\theta - 3a \sin\theta \sqrt{r^2 + r'^2} - \frac{3ar^3 \sin\theta}{\sqrt{r^2 + r'^2}} \right) \delta r \right.$$
$$\left. - \frac{3arr' \sin\theta}{\sqrt{r^2 + r'^2}} \delta r' \right\} d\theta.$$

The condition that this vanish gives

$$\lambda - \frac{3arr' \sin\theta}{\sqrt{r^2 + r'^2}} - 3\int_0^\theta \left(r^2 \sin\theta - a \sin\theta \sqrt{r^2 + r'^2} - \frac{ar^3 \sin\theta}{\sqrt{r^2 + r'^2}} \right) d\theta = I(\theta),$$

whence, $\left(r'' = \frac{d^2r}{d\theta^2}\right)$,

$$- \frac{3arr' \cos\theta}{\sqrt{r^2 + r'^2}} - \frac{3ar^3 \sin\theta}{(r^2 + r'^2)^{\frac{3}{2}}} r'' - \frac{3ar'^3 \sin\theta}{(r^2 + r'^2)^{\frac{3}{2}}} r' - 3r^2 \sin\theta$$
$$+ 3a \sin\theta \sqrt{r^2 + r'^2} + \frac{3ar^3 \sin\theta}{\sqrt{r^2 + r'^2}} = \frac{dI}{d\theta},$$

$$- \frac{3arr' \cos\theta}{\sqrt{r^2 + r'^2}} - \frac{3ar^3 \sin\theta}{(r^2 + r'^2)^{\frac{3}{2}}} r'' - \frac{3a \sin\theta \, (r'^4 - r^4 - 2r^2r'^2 - r'^4 - r^4 - r^2r'^2)}{(r^2 + r'^2)^{\frac{3}{2}}}$$
$$- 3r^2 \sin\theta = \frac{dI}{d\theta},$$

or $$- \frac{ar^3 \sin\theta \, (r + r'')}{(r^2 + r'^2)^{\frac{3}{2}}} - \frac{arr' \cos\theta - 3ar^2 \sin\theta}{\sqrt{r^2 + r'^2}} - r^2 \sin\theta = \frac{1}{3} \frac{dI}{d\theta}.$$

Now $$\frac{d}{d\theta} \left\{ \frac{2ar \sin\theta \, (r \sin\theta - r' \cos\theta)}{\sqrt{r^2 + r'^2}} - r^2 \sin^2\theta \right\}$$

$$= - \frac{2ar^3 \sin\theta \cos\theta + 2ar^2r' \sin^2\theta}{(r^2 + r'^2)^{\frac{3}{2}}} r'' - \frac{2ar'^2 \sin\theta \cos\theta - 4arr' \sin^2\theta}{\sqrt{r^2 + r'^2}}$$
$$+ \frac{2ar^2 \sin\theta \, . \, r' \, (r' \cos\theta - r \sin\theta)}{(r^2 + r'^2)^{\frac{3}{2}}} - \frac{2arr' \cos 2\theta - 2ar^2 \sin 2\theta}{\sqrt{r^2 + r'^2}}$$
$$- 2rr' \sin^2\theta - r^2 \sin 2\theta$$

$$= (r \cos\theta + r' \sin\theta) \left\{ - \frac{2ar^2 \sin\theta \, (r + r'')}{(r^2 + r'^2)^{\frac{3}{2}}} \right.$$
$$\left. - \frac{2ar' \cos\theta - 6ar \sin\theta}{(r^2 + r'^2)^{\frac{1}{2}}} - 2r \sin\theta \right\}.$$

So that

$$\frac{2ar\sin\theta\,(r\sin\theta - r'\cos\theta)}{\sqrt{r^2+r'^2}} - r^2\sin^2\theta = \int_0^\theta \frac{2\,(r\cos\theta + r'\sin\theta)}{3r}\,\frac{dI}{d\theta}\,d\theta.$$

A solution of this equation, when $\frac{dI}{d\theta} = 0$, will be obtained from

$$r\sin\theta\sqrt{r^2+r'^2} = 2a\,(r\sin\theta - r'\cos\theta),$$

which is satisfied by

$$r^2 - 2\xi r\cos\theta + \xi^2 = 4a^2.$$

And so the equation

$$\frac{2ar\sin\theta\,(r\sin\theta - r'\cos\theta)}{\sqrt{r^2+r'^2}} - r^2\sin^2\theta = 0,$$

is satisfied by

$$r = \xi\cos\theta \pm \sqrt{4a^2 - \xi^2\sin^2\theta} + I(\theta),$$

where $I(\theta)$ has instantaneous value when $\theta = 0$, or $\theta = \pi$, and produces no effect on the value, either of the solid or the surface of revolution, as may be seen from the circles traced in fig. (64). The investigation of $\delta^2 u$ would, doubtless, show that u is a maximum when r has the above value.

163. PROB. XII. Having given $u \equiv \int_{x_0}^{x_1} y\sqrt{1+p^2}\,dx$,

we have $$\delta u \equiv \int_{x_0}^{x_1} \left\{\sqrt{1+p^2}\,\delta y + \frac{py\delta p}{\sqrt{1+p^2}}\right\} dx,$$

$$\delta^2 u \equiv \int_{x_0}^{x_1} \left\{\frac{2p\delta y\delta p}{\sqrt{1+p^2}} + \frac{y\delta p^2}{(1+p^2)^{\frac{3}{2}}}\right\} dx.$$

If the question had been to find the surfaces of revolution of minimum area, $\sqrt{y^2}$ would have taken the place of y in u, true discontinuity would have shown itself in the differential coefficient of $\sqrt{y^2}$, and $p = I(x)$, $y = 0$, would have made u a minimum.

Resuming, we have $N \equiv \sqrt{1+p^2}$; $P \equiv \dfrac{py}{\sqrt{1+p^2}}$.

So that
$$\lambda + \frac{py}{\sqrt{1+p^2}} - \int_{x_0}^{x} \sqrt{1+p^2}\, dx = I(x),$$

whence
$$\frac{y}{(1+p^2)^{\frac{3}{2}}} \frac{dp}{dx} - \frac{1}{\sqrt{1+p^2}} = \frac{dI}{dx},$$

and
$$-\frac{y}{\sqrt{1+p^2}} = \int p \frac{dI}{dx} dx.$$

If through an interval $\dfrac{dI}{dx}$ vanish, or $\dfrac{y}{\sqrt{1+p^2}} = b$,
$$\frac{dx^2}{dy} = \frac{b^2}{y^2 - b^2},$$

whence
$$y = \tfrac{1}{2} b \left(e^{\frac{x-c}{b}} + e^{-\frac{x-c}{b}}\right).$$

The value of $\delta^2 u$ shows that, when y is positive, u will be a minimum when $(x_1 - x_0)$ is small enough. To the question of making $\delta^2 u$ a minimum by variation of δy, δp, $\dfrac{d\delta p}{dx}$, p, y, being replaced by those functions of x to which our equations lead, we must find that the passage from one of our solutions to the next in the series gives the answer. We shall gain in simplicity, without important loss of generality, in supposing $x_0 + x_1 = 0$, $c = 0$. Now the locus of the ultimate intersections of, b varying,
$$y = \tfrac{1}{2} b \left(e^{\frac{x}{b}} + e^{-\frac{x}{b}}\right),$$

requires
$$0 = e^{\frac{x}{b}} + e^{-\frac{x}{b}} - \frac{x}{b}\left(e^{\frac{x}{b}} - e^{-\frac{x}{b}}\right).$$

Suppose bk to be the positive root of this equation, then $-bk$ is also a root, and for the locus of ultimate intersections,
$$k^2 y^2 = \tfrac{1}{4} x^2 (e^k + e^{-k})^2.$$

Now $$(e^k + e^{-k})^2 = k^2 (e^k + e^{-k})^2 - 4k^2,$$

so that $$y^2 = \frac{x^2}{k^2 - 1}.$$

If the values of y when $x = x_0$, x_1, are such that the curve must cut these lines, $\int_{x_0}^{x_1} y\sqrt{1+p^2}\,dx$ has no minimum value, and $\int_{x_0}^{x_1} \sqrt{y^2(1+p^2)}\,dx$ has its minimum value in assuming $y = I(x)$. In this case the least surface is the surface of the circular discs formed by the extreme ordinates + the evanescent area of the line joining their centres.

164. PROB. XIII. Having given

$$u \equiv \int_{x_0}^{x_1} \left(\frac{d^2f}{dy^2}\delta y^2 + 2\frac{d^2f}{dydp_1}\delta p_1 \delta y + \ldots + \frac{d^2f}{dp_n^2}\delta p_n^2\right) dx,$$

where $y, p_1, \ldots p_n$, are consistent known functions of x, and f is a known function of $y, p_1, \ldots p_n, x$, to determine δy, $\delta p_1, \ldots$ so that u be a maximum or a minimum according as $\frac{d^2f}{dp_n^2}$, supposed invariable in sign, is negative or positive.

$$\delta u \equiv 2\int_{x_0}^{x_1} \left\{\frac{d^2f}{dy^2}\delta y \delta\delta y + \frac{d^2f}{dydp_1}(\delta p_1 \delta\delta y + \delta y \delta\delta p_1) + \ldots + \frac{d^2f}{dp_n^2}\delta p_n \delta\delta p_n\right\} dx,$$

$$\delta^2 u \equiv 2\int_{x_0}^{x_1} \left(\frac{d^2f}{dy^2}\delta\delta y^2 + 2\frac{d^2f}{dydp_1}\delta\delta p_1 \delta\delta y + \ldots + \frac{d^2f}{dp_n^2}\delta\delta p_n^2\right) dx.$$

So that, as long as $(x_1 - x_0)$ is so small that no system of values for δy, $\delta p_1, \ldots$ obtained from $\delta u = 0$, can render u zero, other than $\delta y = \delta p_1 = \ldots = \delta p_n = 0$, (or, which comes to the same thing, so long as the solutions of $\delta u = 0$, δy, $\delta p_1, \ldots \delta p_n$, δp_{n+1} varying, are such that u only vanishes when they all vanish,) the values, obtained from $\delta u = 0$; for δy, $\delta p_1, \ldots$ will make u a maximum if u is negative and a minimum if u is positive. These values will satisfy the equation

$$\frac{d^2f}{dydp_n}\delta y+\frac{d^2f}{dp_1dp_n}\delta p_1+\ldots+\frac{d^2f}{dp_n^{\,2}}\delta p_n$$

$$-\frac{d^{-1}}{dx^{-1}}\left(\frac{d^2f}{dydp_{n-1}}\delta y+\frac{d^2f}{dp_1dp_{n-1}}\delta p_1+\ldots+\frac{d^2f}{dp_{n-1}^{\,2}}\delta p_{n-1}+\frac{d^2f}{dp_{n-1}dp_n}\delta p_n\right)$$

$$+\ldots+(-1)^{n-1}\frac{d^{-n+1}}{dx^{-n+1}}\left(\frac{d^2f}{dydp_1}\delta y+\frac{d^2f}{dp_1^{\,2}}\delta p_1+\ldots+\frac{d^2f}{dp_1dp_n}\delta p_n\right)$$

$$+(-1)^n\frac{d^{-n}}{dx^{-n}}\left(\frac{d^2f}{dy^2}\delta y+\frac{d^2f}{dydp_1}\delta p_1+\ldots+\frac{d^2f}{dydp_n}\delta p_n\right)=I(x).$$

Now if f satisfy the equation

$$\frac{df}{dp_n}-\frac{d^{-1}}{dx^{-1}}\frac{df}{dp_{n-1}}+\frac{d^{-2}}{dx^{-2}}\frac{df}{dp_{n-2}}-\ldots+(-1)^{n-1}\frac{d^{-n+1}}{dx^{-n+1}}\frac{df}{dp_1}$$

$$+(-1)^n\frac{d^{-n}}{dx^{-n}}\frac{df}{dy}=I_1(x),$$

and if, either no variation being given to $I_1(x)$, or a variation being given, yet such that $I_1+\delta I_1$ still possesses the character characteristic of the functions which we denote by $I(x)$, δy, δp_1, ... be consistent functions, and δf such that initially

$$\delta f=\frac{df}{dy}\delta y+\frac{df}{dp_1}\delta p_1+\ldots+\frac{df}{dp_n}\delta p_n,$$

and
$$\frac{d(f+\delta f)}{dp_n}-\frac{d^{-1}}{dx^{-1}}\frac{d(f+\delta f)}{dp_{n-1}}+\ldots$$

$$+(-1)^n\frac{d^{-n}}{dx^{-n}}\frac{d(f+\delta f)}{dy}=I_1(x)+\delta I_1(x),$$

then the equation obtained from $\delta u=0$ is satisfied. And conversely, if the equation obtained from $\delta u=0$ is satisfied, both f and $(f+\delta f)$ satisfy the condition that δv vanish when $v\equiv\int_{x_0}^{x_1}fdx$.

Be it observed that if $\frac{d^2f}{dp_n^2}$ vanish identically, f can only involve p_n in the first power, and, F being a function of x, y, p_1, … p_{n-1},

$$f+\frac{dF}{dx}=\phi(x, y, \dots p_{n-1})+p_n\left(\frac{df}{dp_n}+\frac{dF}{dp_{n-1}}\right),$$

$\frac{df}{dp_n}+\frac{dF}{dp_{n-1}}$ can be always made zero by proper choice of F.

165. PROB. XIV. Having given $u\equiv\int_{x_0}^{x_1}\frac{y^n p_2}{p_1}dx$, to find when u is a maximum or a minimum.

$$\delta u=\int_{x_0}^{x_1}\left(\frac{ny^{n-1}p_2}{p_1}\delta y-\frac{y^n p_2}{p_1^2}\delta p_1+\frac{y^n}{p_1}\delta p_2\right)dx,$$

assume $\frac{dF}{dp_1}=-\frac{d}{dp_2}\left(\frac{y^n p_2}{p_1}\right)=-\frac{y^n}{p_1}$; $F=-y^n\log\sqrt{p_1^2}$.

Then

$$u+\Big|_{x_0}^{x_1}F=\int_{x_0}^{x_1}\left(\frac{y^n p_2}{p_1}-ny^{n-1}p_1\log\sqrt{p_1^2}-\frac{y^n p_2}{p_1}\right)dx$$

$$=-n\int_{x_0}^{x_1}y^{n-1}p_1\log\sqrt{p_1^2}\,dx.$$

If, when $x=x_0$, and when $x=x_1$, y, p_1 are given, $\Big|_{x_0}^{x_1}F$ is not susceptible of variation. But if they are not given, without altering $\int_{x_0}^{x_1}y^{n-1}p_1\log\sqrt{p_1^2}\,dx$, u can be changed in value by altering p_1 at the limits, y (not being zero) remaining unaltered. Now $\Big|_{x_0}^{x_1}F$ does not admit of a maximum or a minimum value, since $\log\sqrt{p_1^2}$ can have any value from $-\infty$ through zero to $+\infty$. So that u cannot have a maximum or a minimum value unless $\Big|_{x_0}^{x_1}F$ is a constant, or otherwise restricted in value. When $\Big|_{x_0}^{x_1}F$ is a constant, u will be a maximum or a minimum when

$-\int_{x_0}^{x_1} y^{n-1} p_1 \log\sqrt{p_1^2}\,dx$ is a maximum or a minimum. So we may confine our attention to this integral, in which the values of y, p_1 at the limits are supposed given.

We have, λ being an unknown arbitrary constant,

$$\delta\left(u + \Big|_{x_0}^{x_1} F\right) = -n\int_{x_0}^{x_1}\{(n-1)\,y^{n-2}p_1 \log\sqrt{p_1^2}\,\delta y + y^{n-1}\log\sqrt{p_1^2}\,\delta p_1 + y^{n-1}\,\delta p_1 + \lambda\delta p_1\}\,dx.$$

And, in $\delta^2\left(u + \Big|_{x_0}^{x_1} F\right)$, the coefficient of δp_1^2 is $-\frac{ny^{n-1}}{p_1}$, which must always be positive when u is a minimum, and negative when u is a maximum. $\delta u = 0$ gives

$$\lambda + y^{n-1}\log\sqrt{p_1^2} + y^{n-1} - \frac{d^{-1}}{dx^{-1}}\{(n-1)\,y^{n-2}p_1 \log\sqrt{p_1^2}\} = I(x),$$

and
$$\frac{y^{n-1}p_2}{p_1} + (n-1)\,y^{n-2}p_1 = \frac{dI}{dx},$$

$$y^{n-1}p_1 = \int p_1 \frac{dI}{dx}\,dx,$$

whence
$$y^n = n\iint p_1 \frac{dI}{dx}\,dx^2.$$

When $I(x)$ is other than zero only when $x = x_0$ or $x = x_1$, we shall have, between these limits,

$$y^n = c_1 x + c_2, \qquad ny^{n-1}p_1 = c_1.$$

u cannot be a maximum unless c_1 be positive, nor a minimum unless c_1 be negative. We shall confine our attention to the case when c_1 is positive. $\delta^2 u$ can be, in the usual way, thus transformed,

$$-\frac{\delta^2 u}{n} = \int_{x_0}^{x_1}\frac{y^{n-1}}{p_1}\left\{\delta p_1 + \frac{(n-1)\,p_1}{y}\,\delta y\right\}^2 dx + \Big|_{x_0}^{x_1}\{(n-1)\,y^{n-2}\log\sqrt{p_1^2}\,\delta y^2\}$$
$$-\int_{x_0}^{x_1}\left\{(n-1)^2 p_1 y^{n-3}\,\delta y^2 + \frac{(n-1)\,y^{n-2}p_2}{p_1}\,\delta y^2\right\}dx.$$

So $\delta^2 u$ is invariable in sign, vanishing, however, when $\frac{y^{n-1}}{p_1}$ vanishes. Therefore, if on the curve for which

$$y^n = c_1x + c_2 + I_1(x), \qquad ny^{n-1}p_1 = c_1 + \frac{dI_1}{dx},$$

y, p_1, have the given values when $x = x_0$ and $x = x_1$, and $\left(c_1 + \frac{dI}{dx}\right)$ is always positive, for this curve u is a maximum.

Suppose (fig. 65) $B_0P_0P_1B_1$ is a curve throughout which the above equations are satisfied, X_0B_0, X_1B_1, being the given values of y at the limits. Then, as x changes from the abscissa of B_0 to the abscissa of P_0,

$$\int_{x_0}^{x_0+} y^{n-1}p_1 \log\sqrt{p_1^2}\,dx > L\int_{x_0}^{x_0+} y^{n-1}p_1\,dx > \frac{L}{n}(X_0P_0^n - X_0B_0^n),$$

whatever finite numerical magnitude L may be. So among all such curves the greatest value is given to u by that for which $B_0P_0 = B_1P_1 = 0$, all others making $u = -\infty$. Observe that when y is invariable in sign throughout p must be invariable in sign throughout.

We can show that when X_0B_0, X_1B_1, do not vanish, there is no maximum maximorum among the curves for which δu vanishes. For in the curve B_0PB_1, where B_0P is small but finite, u for PB_1 differs from u for B_0B_1 by a finite quantity, and u for B_0P is infinitely great and positive. So that, while u for B_0B_1 is finite, u for B_0PB_1 is infinitely great and positive.

The condition that $y^{n-1}p_1$ has invariable sign excludes all curves which, starting from a point in the ordinate through X_0 other than X_0, cut X_0X_1 before reaching the ordinate through X_1.

There remains to be considered the case of

$$p_1 = 0, \quad y = 0.$$

In this case some differential coefficients of y^{n-1} are truly discontinuous unless n is a positive integer. We shall have

$$\delta\left(u + \Big|_{x_0}^{x_1} F\right) = -n\int_{x_0}^{x_1}(\delta y)^{n-1}\delta p_1 \log\sqrt{\delta p_1^2}\,dx.$$

If $(n-1)$ is such that $(\delta y)^{n-1}$ is always positive and real, *i. e.*, if, when $(n-1)$ is expressed as a vulgar fraction in its lowest terms, its numerator is even, a change in the sign of δp_1 will change the sign of $\delta\left(u+\Big|_{x_0}^{x_1}F\right)$; so in this case we have neither a maximum nor a minimum.

In fact in all cases if we take, δc_0, δc_1, being of the same sign and positive, if necessary,

$$\delta p_1 = \delta c_0, \text{ from } x = x_0 \text{ till } x = \frac{x_1+x_0}{2}+k,$$

$$\delta p_1 = -\delta c_1, \text{ from } x = \frac{x_1+x_0}{2}+k \text{ till } x = x_1;$$

we shall have

$$\delta y = \delta c_0 (x - x_0), \text{ from } x = x_0 \text{ till } x = \frac{x_1+x_0}{2}+k,$$

$$\delta y = \delta c_1 (x_1 - x), \text{ from } x = \frac{x_1+x_0}{2}+k \text{ till } x = x_1:$$

and we must have, to prevent true discontinuity,

$$\delta c_0 \left(\frac{x_1-x_0}{2}+k\right) = \delta c_1 \left(\frac{x_1-x_0}{2}-k\right)$$

$$= \frac{2k\delta c_1 \delta c_0}{\delta c_1 - \delta c_0} = \frac{(x_1-x_0)\,\delta c_1 \delta c_0}{\delta c_1 + \delta c_0}.$$

The expression of δp_1 by a Fourierian definite integral will round the sharp angles but produce no effect on the value of $\delta\left(u+\Big|_{x_0}^{x_1}F\right)$. Then will

$$\delta\left(u+\Big|_{x_0}^{x_1}F\right) = -n\left(\int_{x_0}^{\frac{x_1+x_0}{2}+k} + \int_{\frac{x_1+x_0}{2}+k}^{x_1}\right)(\delta y)^{n-1}\,\delta p_1 \log\sqrt{\delta p_1^{\,2}}\,dx$$

$$= -n\left\{\int_{x_0}^{\frac{x_1+x_0}{2}+k} \delta c_0^{\,n} \log\sqrt{\delta c_0^{\,2}}\,(x-x_0)^{n-1}\,dx\right.$$

$$\left. - \int_{\frac{x_1+x_0}{2}+k}^{x_1} (\delta c_1)^n \log\sqrt{\delta c_1^{\,2}}\,(x_1-x)^{n-1}\,dx\right\}$$

$$= -\delta c_0^n \log\sqrt{\delta c_0^2}\left(\frac{x_1 - x_0}{2} + k\right)^n$$

$$+ (\delta c_1)^n \log\sqrt{\delta c_1^2}\left(\frac{x_1 - x_0}{2} - k\right)^n$$

$$= \delta c_0^n \left(\frac{x_1 - x_0}{2} + k\right)^n (\log\sqrt{\delta c_1^2} - \log\sqrt{\delta c_0^2}),$$

and changes sign with a change in sign of k. So when $p_1 = y = 0$ we have neither a maximum nor a minimum.

Our conclusion is that, in all cases, the solutions, infinite in number, given by the method of the Variations Calculus, give to u values ranging from $-\infty$ to $+\infty$, and that none of these is ever a maximum or a minimum when, by Differential Calculus methods, these various solutions are compared between themselves.

166. PROB. XV. Having given $u \equiv \int_{x_0}^{1} p_n^m dx$,

find y so that u be a maximum or a minimum.

Here $P_n \equiv mp_2^{m-1}$; $\dfrac{dP_2}{dp_n} = m(m-1)p_n^{m-2}$.

If the numerator and denominator of m are both odd, u can be neither a maximum nor a minimum; but if either are even u cannot be negative, and

$$p_n = I(x)$$

makes u a minimum when m is positive. The problem in no respect differs from that of finding y when $\int_{x_0}^{x_1} y^m dx$ is a maximum or a minimum. When m is not a positive integer true discontinuity appears, when $p_n = 0$, in p_n^m when m is negative, and in some of the successive differential coefficients of p_n^m when m is positive.

167. Note on Art. 126. Suppose it is required to determine the forms of the functions y, z, so that

$$u \equiv \int_{x_0}^{x_1} f\left(x, y, \frac{dy}{dx}, \dots \frac{d^n y}{dx^n}, z, \frac{dz}{dx}, \dots \frac{d^n z}{dx^n}\right) dx,$$

or, as we may conveniently write it,

$$u \equiv \int_{x_0}^{x_1} f(x, y, p_1, \dots p_n, z, q_1, \dots q_n)\, dx$$

be a maximum or a minimum, when $x, y, p_1, \dots q_n$ are connected by an equation,

$$0 = L(x, y, p_1, \dots q_n).$$

We suppose that y, z, and their differential coefficients may not be truly discontinuous, and that generally the successive differential coefficients of f and L, with respect to $x, y, \dots q_n$, are not truly discontinuous. As in Art. 122, cases when true discontinuity occurs should be separately examined in order to determine what then happens.

If $y, z, p_1, \dots q_n$ be such that for all values of $\delta y, \delta z, \delta p_1, \dots \delta q_n$, $(u + \int_{x_0}^{x_1} XL dx)$, X being some finite function of x, is a maximum or a minimum, it is evident that, in the more restricted case when $L = 0$, and $\delta y, \delta z, \delta p_1, \dots \delta q_n$ satisfy $\delta L = 0$, $(u + \int_{x_0}^{x_1} XL dx)$, and therefore u, will be a maximum or a minimum.

But if $y, z, p_1, \dots q_n$, be such that $L = 0$, and by proper choice of $\delta y, \delta z, \delta p_1, \dots \delta q_n$,

$$\int_{x_0}^{x_1} f(x, y + \delta y, p_1 + \delta p_1, \dots q_n + \delta q_n)\, dx,$$

$$\int_{x_0}^{x_1} f(x, y - \delta y, p_1 - \delta p_1, \dots q_n - \delta q_n)\, dx,$$

have opposite signs initially, and yet the equations,

$$\begin{aligned} 0 &= L(x, y + \delta y, p_1 + \delta p_1, \dots q_n + \delta q_n) \\ &= L(x, y - \delta y, p_1 - \delta p_1, \dots q_n - \delta q_n), \end{aligned}$$

are satisfied, then $\delta\,(u+\int_{x_0}^{x_1} XLdx)$ can, by a proper choice of δy, δz, δp_1, ... δq_n, be made to change signs when they change signs. Therefore, u cannot be a maximum or a minimum unless $\delta\,(u+\int_{x_0}^{x_1} XLdx)$ vanish, or be truly discontinuous and changing sign. So not only solutions of the question, 'When is $(u+\int_{x_0}^{x_1} XLdx)$ a maximum or a minimum?' but also solutions of the question, 'When does $\delta\,(u+\int_{x_0}^{x_1} XLdx)$ vanish?' may make u a maximum or a minimum. That is to say, when $\delta^2 (u+\int_{x_0}^{x_1} XLdx)$ may be made to vanish or change sign by proper choice of δy, δz, δp_1, ... δq_n, although $\delta\,(u+\int_{x_0}^{x_1} XLdx)$ vanish, such proper choice, through being inconsistent with $L=0$, may fail to be applicable when the question is considered of making u a maximum or a minimum when $L=0$.

168. Again, suppose it is required to determine the forms of the function y, so that,

$$u \equiv \int_{x_0}^{x_1} f(x,\ y,\ p_1,\ \ldots . p_n)\, dx,$$

is a maximum or a minimum, and yet there exists the condition,

$$c=\int_{x_0}^{x_1} F(x,\ y,\ p_1,\ \ldots\ p_n)\, dx.$$

Barring, as in Arts. 122 and 167, true discontinuity, it will appear that if $\left\{u+\lambda\left(\int_{x_0}^{x_1} Fdx - c\right)\right\}$, where λ is some finite constant, is a maximum or a minimum, it will be so also in the more restricted case when $c=\int_{x_0}^{x_1} Fdx$.

Also if

$$\int_{x_0}^{x_1} f(x,\ y+\delta y,\ \ldots\ldots\ p_n+\delta p_n)\, dx,$$

$$\int_{x_0}^{x_1} f(x,\ y-\delta y,\ \ldots\ldots\ p_n-\delta p_n)\, dx,$$

have different signs initially, notwithstanding that

$$c=\int_{x_0}^{x_1} F(x,\ y+\delta y,\ \ldots\ldots\ p_n+\delta p_n)\, dx$$

$$=\int_{x_0}^{x_1} F(x,\ y-\delta y,\ \ldots\ldots\ p_n-\delta p_n)\, dx,$$

then shall $\delta\left\{u+\lambda\left(\int_{x_0}^{x_1} Fdx-c\right)\right\}$ change sign with δy. So not only solutions of the question, 'When is $(u+\lambda\int_{x_0}^{x_1} Fdx)$ a maximum or a minimum?' but also solutions of the question, 'When does $\delta\,(u+\lambda\int_{x_0}^{x_1} Fdx)$ vanish?' may make u a maximum or a minimum when $c=\int_{x_0}^{x_1} Fdx$. That is to say, when $\delta^2\,(u+\lambda\int_{x_0}^{x_1} Fdx)$ may be made to vanish or change sign by proper choice of δy, δp_1, ..., such proper choice, through being inconsistent with $\int_{x_0}^{x_1} \delta F.\, dx=0$, may be inapplicable in the question of making u a maximum or a minimum, when $c=\int_{x_0}^{x_1} Fdx$. It will be clear that, in such a case, the sign of $\frac{d^2 f}{dp_n^{\,2}}$ will determine whether any particular form of the function y, obtained from $\delta\,(u+\lambda\int_{x_0}^{x_1} Fdx)=0$, renders u a maximum or a minimum.

169. **Prob. XVI.** Having given

$$u\equiv\int_{x_0}^{x_1} \sqrt{1+p_1^{\,2}+q_1^{\,2}}\, dx,$$

$$0=x+yp_1+zq_1,$$

$$y_1 - y_0 = \int_{x_0}^{x_1} p_1 dx,$$

$$z_1 - z_0 = \int_{x_0}^{x_1} q_1 dx,$$

to determine what functions y and z are of x when u is a minimum.

If we determine y, z, so that v be a minimum, where

$$v \equiv \int_{x_0}^{x_1} \{\sqrt{1+p_1^2+q_1^2} + X(x+yp_1+zq_1) + \lambda_1 p_1 + \mu_1 q_1\}\, dx,$$

we find

$$\left.\begin{aligned} \lambda_1 + \frac{p_1}{\sqrt{1+p_1^2+q_1^2}} + Xy - \frac{d^{-1}Xp_1}{dx^{-1}} &= I_p(x), \\ \mu_1 + \frac{q_1}{\sqrt{1+p_1^2+q_1^2}} + Xz - \frac{d^{-1}Xq_1}{dx^{-1}} &= I_q(x), \end{aligned}\right\} \ldots\ldots\ldots (1),$$

and $\delta^2 v$, it will be seen, will be always positive. The equations lead to

$$\frac{d}{dx}\frac{p_1}{\sqrt{1+p_1^2+q_1^2}} + y\frac{dX}{dx} = \frac{dI_p}{dx},$$

$$\frac{d}{dx}\frac{q_1}{\sqrt{1+p_1^2+q_1^2}} + z\frac{dX}{dx} = \frac{dI_q}{dx},$$

whence $$\frac{p_1 z - q_1 y}{\sqrt{1+p_1^2+q_1^2}} = \int\left(z\frac{dI_p}{dx} - y\frac{dI_q}{dx}\right) dx.$$

Also, since $0 = x + yp_1 + zq_1$,

$$\begin{aligned}(p_1 z - q_1 y)^2 &= (p_1^2+q_1^2)(z^2+y^2) - (p_1 y + q_1 z)^2 \\ &= (p_1^2+q_1^2)(c^2-x^2) - x^2 \\ &= (p_1^2+q_1^2)\, c^2 - (1+p_1^2+q_1^2)\, x^2,\end{aligned}$$

$$\frac{p_1 z - q_1 y}{\sqrt{1+p_1^2+q_1^2}} = \pm\sqrt{\frac{(p_1^2+q_1^2)\, c^2}{1+p_1^2+q_1^2} - x^2}.$$

So that during an interval in which $\int\left(z\frac{dI_p}{dx} - y\frac{dI_q}{dx}\right)dx$ has contant value β, we must have

$$(\beta^2 + x^2)(1 + p_1^2 + q_1^2) = (p_1^2 + q_1^2)c^2,$$

$$1 + p_1^2 + q_1^2 = \frac{c^2}{c^2 - \beta^2 - x^2},$$

$$p_1 z - q_1 y = \beta\sqrt{\frac{c^2}{c^2 - \beta^2 - x^2}},$$

whence can be obtained $\tan^{-1}\frac{y}{z}$ by integration. But the same result may be obtained with more facility, and no sacrifice of generality, in observing that if

$$x + yB + zC = 0, \qquad x^2 + y^2 + z^2 = c^2,$$

we have $\quad 1 + p_1 B + q_1 C = 0, \qquad x + yp_1 + zq_1 = 0,$

whence $\quad p_1 = \frac{z - Cx}{Cy - Bz}, \qquad q_1 = \frac{Bx - y}{Cy - Bz},$

$$p_1^2 + q_1^2 = \frac{z^2 + y^2 - 2(Cz + By)x + (C^2 + B^2)x^2}{(Cy - Bz)^2}$$

$$= \frac{c^2 - x^2 + 2x^2 + (C^2 + B^2)x^2}{(B^2 + C^2)(c^2 - x^2) - x^2},$$

for $\quad (Cy - Bz)^2 = (B^2 + C^2)(y^2 + z^2) - (Cz + By)^2,$

$$p_1^2 + q_1^2 = \frac{\beta^2 + x^2}{c^2 - \beta^2 - x^2},$$

if $\quad \beta^2 = \frac{c^2}{1 + B^2 + C^2}.$

The geometrical interpretation of this is that the great circle joining two points on a sphere is the shortest line on the sphere that can be drawn from one point to the other. The effect of $I_p(x)$ and $I_q(x)$ having at any instant value is to obtain as a solution a curve wherein for that value of x a finite portion of the small circle parallel plane Oyz lies on the curve.

170. **Prob. XVII.** To find a curve of given sufficient length, joining two given points, such that, if it be made to revolve about a given line, the superficial area of the generated surface may be a maximum or a minimum.

Taking the axis of revolution for the axis of x, we must have $\int_{x_0}^{x_1} y\sqrt{1+p^2}\,dx$ a maximum or a minimum when $\int_{x_0}^{x_1}\sqrt{1+p^2}\,dx$, $\int_{x_0}^{x_1} p\,dx$ are given. So we must enquire concerning the maxima and minima values of

$$u \equiv \int_{x_0}^{x_1} \{(y+\lambda)\sqrt{1+p^2} + \mu p\}\,dx,$$

whence $$\delta u \equiv \int_{x_0}^{x_1} \{\sqrt{1+p^2}\,\delta y + \frac{p\,(y+\lambda)}{\sqrt{1+p^2}}\,\delta p + \mu\delta p\}\,dx,$$

$$\delta^2 u \equiv \int_{x_0}^{x_1} \left\{\frac{2p\delta y\delta p}{\sqrt{1+p^2}} + (y+\lambda)\frac{\delta p^2}{(1+p^2)^{\frac{3}{2}}}\right\}dx,$$

and from $$\mu + \frac{p\,(y+\lambda)}{\sqrt{1+p^2}} - \frac{d^{-1}}{dx^{-1}}\sqrt{1+p^2} = I(x),$$

we have $$\frac{p^2}{\sqrt{1+p^2}} + (y+\lambda)\frac{d}{dx}\left(\frac{p}{\sqrt{1+p^2}}\right) - \sqrt{1+p^2} = \frac{dI}{dx},$$

$$+\frac{(y+\lambda)\,p}{(1+p^2)^{\frac{3}{2}}}\frac{dp}{dx} - \frac{p}{\sqrt{1+p^2}} = p\frac{dI}{dx},$$

$$-\frac{y+\lambda}{\sqrt{1+p^2}} = \int p\frac{dI}{dx}\,dx.$$

For an interval during which $\int p\frac{dI}{dx}\,dx$ is constant,

$$y+\lambda = \tfrac{1}{2}b\left(e^{\frac{x+h}{b}} + e^{-\frac{x+h}{b}}\right).$$

For the length of this curve we have

$$l=\int_{x_0}^{x_1}\sqrt{1+p^2}\,dx=\int_{x_0}^{x_1}\frac{y+\lambda}{b}\,dx=\Big|_{x_0}^{x_1}\frac{b}{2}\left(e^{\frac{x+h}{b}}-e^{-\frac{x+h}{b}}\right),$$

$$\therefore\ l=\sqrt{(y_1+\lambda)^2-b^2}+\sqrt{(y_0+\lambda)^2-b^2},$$

when the abscissa of the point for which $\frac{dy}{dx}$ vanishes $> x_0$ and $< x_1$: in other cases the sum of these surds must be replaced by their difference. Also we have

$$e^{\frac{x_1+h}{b}}=\frac{y_1+\lambda}{b}\pm\frac{\sqrt{(y_1+\lambda)^2-b^2}}{b},\quad e^{\frac{x_0+h}{b}}=\frac{y_0+\lambda}{b}\pm\frac{\sqrt{(y_0+\lambda)^2-b^2}}{b},$$

the upper or lower signs being taken according as $\frac{x_1+h}{b}$, $\frac{x_0+h}{b}$, are positive or negative respectively. It will be perceived that if h, λ, are eliminated, and b' be a value which satisfies the resulting equation when substituted for b, that $(-b')$ will likewise satisfy the equation. One will give a maximum, and the other a minimum, solution. In the particular case of symmetry, when $y_1=y_0$, the equations solve with facility, and we have,

$$l=2\sqrt{(y_1+\lambda)^2-b^2},\quad \frac{x_1+x_0}{2}+h=0,$$

$$e^{\frac{x_1-x_0}{2b}}=\frac{y_1+\lambda}{b}+\frac{\sqrt{(y_1+\lambda)^2-b^2}}{b}$$

$$=\frac{y_1+\lambda}{b}+\frac{l}{2b},$$

$$e^{\frac{x_1-x_0}{b}}-\frac{l}{b}e^{\frac{x_1-x_0}{2b}}-1=0.$$

$b=+\infty$ makes the left-hand side

$$=1+\frac{x_1-x_0}{b}+\ldots-\frac{l}{b}-\ldots-1=\frac{x_1-x_0-l}{b}-\ldots$$

which is negative, and $b=0$ makes it positive, so that there is a positive root, that root with sign changed likewise satisfies the equation. And $(y+\lambda)$ has always the same sign as b. So there is always one maximum, and one minimum solution. A solution which requires y to change sign will not correspond to the geometrical question proposed. In that case we must enquire, as in Prob. XIII., concerning the minimum value of

$$\int_{x_0}^{x_1}\{(\sqrt{y^2}+\lambda)\sqrt{1+p^2}+\mu p\}\,dx.$$

171. PROB. XVIII. Among all the curves joining two given points throughout which the relation, $y+z=s$, holds, to determine the ones for which $\int_{x_0}^{x_1}(y+z)\,dx$ is a maximum or a minimum.

We shall suppose $x_0=y_0=z_0=s_0=0$.

We have $$(p_1+q_1)^2=1+p_1^2+q_1^2,$$

or $$2p_1q_1=1.$$

So we must enquire concerning the maxima and minima values of u, where

$$u\equiv\int_{x_0}^{x_1}\{y+z+X(1-2p_1q_1)+\mu p_1+\nu q_1\}\,dx,$$

$$\delta u\equiv\int_{x_0}^{x_1}\{\delta y+\delta z-2X(p_1\delta q_1+q_1\delta p_1)+\mu\delta p_1+\nu\delta q_1\}\,dx,$$

$$\delta^2 u\equiv\int_{x_0}^{x_1}(-4X\delta p_1\delta q_1)\,dx.$$

We find $$\mu-2Xq_1-\frac{d^{-1}1}{dx^{-1}}=I_p(x),$$

$$\nu-2Xp_1-\frac{d^{-1}1}{dx^{-1}}=I_q(x);$$

whence $$\mu+c'_1-2Xq_1-x=I_p(x),$$

$$\nu+c'_2-2Xp_1-x=I_q(x);$$

and $$\frac{\mu + c'_1 - x - I_p}{\nu + c'_2 - x - I_q} = \frac{q_1}{p_1} = 2q_1^2 = \frac{1}{2p_1^2}.$$

So that $$\sqrt{2}\frac{dy}{dx} = \sqrt{\frac{\nu + c'_2 - x - I_q}{\mu + c'_1 - x - I_p}}.$$

I_p, I_q will have no effect upon the values of y, z, except when either $(\mu + c_1 - x - I_p)$ or $(\nu + c_2 - x - I_q)$ vanish. In that case they may have effect even if they vanish (compare Arts. 93, 94). So we may neglect them, reserving for further consideration what may happen upon the evanescence of either p_1 or q_1. Also we may put, $\nu + c'_2 = c_2$, $\mu + c'_1 = c_1$, so that

$$\sqrt{2}\frac{dy}{dx} = \sqrt{\frac{c_2 - x}{c_1 - x}}, \quad \sqrt{2}\frac{dz}{dx} = \sqrt{\frac{c_1 - x}{c_2 - x}}.$$

From this can be obtained, A, B, being constants, and c_1, c_2, positive,

$$\sqrt{2}(y + z + A) = -2\sqrt{(c_1 - x)(c_2 - x)},$$

$$\sqrt{2}(y - z + B) = (c_2 - c_1)\log\left\{\frac{c_1 + c_2 - 2x}{2} - \sqrt{(c_1 - x)(c_2 - x)}\right\}.$$

As we suppose that y, z, s vanish when $x = 0$, we have

$$\sqrt{2}(y_1 + z_1) = 2\sqrt{c_1 c_2} - 2\sqrt{(c_1 - x_1)(c_2 - x_1)},$$

$$\sqrt{2}(y_1 - z_1) = (c_2 - c_1)\log\frac{c_1 + c_2 - 2x_1 - 2\sqrt{(c_1 - x_1)(c_2 - x_1)}}{c_1 + c_2 - 2\sqrt{c_1 c_2}},$$

two equations to determine c_1, c_2. Observe that we must have throughout,

$$s^2 > x^2 + y^2 + z^2, \quad 2yz > x^2,$$

$$y + z > 2\sqrt{yz} > \sqrt{2}x.$$

The point (x_1, y_1, z_1) must therefore lie in the cup of the right circular cone $(2yz = x^2)$. Also the above logarithm is positive, so that $(c_2 - c_1)$ and $(y_1 - z_1)$ have the same sign. The surface

($2yz = x^2$) has the axes of y and of z for generating lines, and the line ($x = 0$, $y = z$) for axis. The value of $\delta^2 u$ shows that u will be a maximum if X throughout negative. For p_1, q_1 have always the same sign, and, $p_1\delta q_1 + q_1\delta p_1$ vanishing, δp_1 and δq_1 must have different signs. Hence u is a maximum or a minimum according as c_1 is negative or positive.

Suppose, firstly, $2y_1z_1 = x_1^2$. Our equations will shew that in this case u is both a maximum and a minimum, in fact, is conjugate or isolate in character. This should be; for $(y_1 + z_1)$ is in this case the length of the straight line joining (x_1, y_1, z_1) to the origin, and no variation can be assigned so that the magnitude of s shall not be increased.

In this case our equations give,

$$\frac{\sqrt{2}\,(y_1 - z_1)}{c_2 - c_1} = \log\frac{c_1 + c_2 - 2\sqrt{2y_1z_1} + \sqrt{2}\,(y_1 + z_1) - 2\sqrt{c_1c_2}}{c_1 + c_2 - 2\sqrt{c_1c_2}}$$

$$= \log\left\{1 + \sqrt{2}\left(\frac{\sqrt{y_1} - \sqrt{z_1}}{\sqrt{c_2} - \sqrt{c_1}}\right)^2\right\},$$

and $\quad 2\,(y_1 + z_1)^2 - 4\sqrt{2c_1c_2}\,(y_1 + z_1) = 8y_1z_1 - 4\,(c_1 + c_2)\sqrt{2y_1z_1},$

or $\sqrt{2}\,(y_1 - z_1)^2 = 4\,\{\sqrt{c_1c_2}\,(y_1 + z_1) - \sqrt{y_1z_1}\,(c_1 + c_2)\}$

$$= (\sqrt{c_2} + \sqrt{c_1})^2(\sqrt{y_1} - \sqrt{z_1})^2 - (\sqrt{y_1} + \sqrt{z_1})^2(\sqrt{c_2} - \sqrt{c_1})^2.$$

Put
$$\sqrt{y_1} - \sqrt{z_1} = (\sqrt{c_2} - \sqrt{c_1})\,\phi,$$
$$\sqrt{y_1} + \sqrt{z_1} = (\sqrt{c_2} + \sqrt{c_1})\,\psi.$$

Then
$$\sqrt{2}\,.\,\phi\psi = \log\,(1 + \sqrt{2}\,.\,\phi^2),$$
$$\sqrt{2}\phi^2\psi^2 = \phi^2 - \psi^2,$$
$$\psi^2 = \frac{\phi^2}{1 + \sqrt{2}\phi^2},$$
$$\frac{\sqrt{2}\,.\,\phi^2}{1 + \sqrt{2}\phi^2} = \log\,(1 + \sqrt{2}\phi^2),$$
$$(1 + \sqrt{2}\,.\,\phi^2) - 1 = (1 + \sqrt{2}\phi^2)\log\,(1 + \sqrt{2}\phi^2).$$

Now if we differentiate $\left(\log z + \frac{1}{z} - 1\right)$ we obtain $\left(\frac{1}{z} - \frac{1}{z^2}\right)$, which is always positive when $z > 1$. So this equation has no real root except $\phi = 0$, whence also $\frac{\psi}{\phi} = 1$.

Now, if $\sqrt{2}y_1 = \sqrt{2}z_1 = x_1$, $c_1 = c_2$ and is perfectly indeterminate.

But if $y_1 \mp z_1$, c_2, c_1, are both infinite, and

$$\frac{\sqrt{c_2} - \sqrt{c_1}}{\sqrt{c_2} + \sqrt{c_1}} = \frac{\sqrt{y_1} - \sqrt{z_1}}{\sqrt{y_1} + \sqrt{z_1}},$$

so that
$$\frac{c_2}{c_1} = \frac{y_1}{z_1},$$

whence
$$\sqrt{2}y = \sqrt{\frac{y_1}{z_1}}\,x, \quad \sqrt{2}z = \sqrt{\frac{z_1}{y_1}}\,x,$$

the equation to the straight line joining (x_1, y_1, z_1) with the origin. If then (x_1, y_1, z_1) lie on the cone, $2yz = x^2$, the straight line joining it with the origin gives to u a conjugate, that is an isolate, value.

Suppose, secondly, $\sqrt{2}\,y_1 = \sqrt{2}\,z_1 > x_1$. It is impossible that the equations which determine c_1, c_1, should be satisfied by any finite values. If through the point P (fig. 66), whose co-ordinates are (x_1, y_1, z_1), be drawn lines parallel to the axis of y and the axis of z, piercing the cone in Q_1 and below P_1, which last points are to be joined with the origin, we obtain in these broken lines a solution, which will be a conjugate one, for the same reason that the solution in our first case was a conjugate one, but only with reference to all lines which coincide with it throughout when p_1, or q_1, is infinite, the only lines with which in the Variations Calculus it can be compared.

Suppose, thirdly, $y_1 > z_1$, $2y_1z_1 > x_1^2$, $c_1 = x_1$. This will require that

$$(y_1 + z_1)^2 = 2c_2x_1,$$

and
$$\frac{2\sqrt{2}\,(y_1 - z_1)\,x_1}{(y_1 + z_1)^2 - 2x_1^2} = \log\frac{y_1 + z_1 + \sqrt{2}\,x_1}{y_1 + z_1 - \sqrt{2}\,x_1}.$$

In this case u is rendered a minimum by the solution

$$\sqrt{2}\,(y+z) = \sqrt{2}\,(y_1+z_1) - 2\sqrt{(x_1-x)\,(c_2-x)},$$

$$2\sqrt{2}\,(y-z)\,x_1 = \{(y_1+z_1)^2 - 2x_1^2\}$$

$$\log\frac{2x_1^2 + (y_1+z_1)^2 - 4x_1x - 2\sqrt{2}\,\sqrt{x_1\,(x_1-x)\{(y_1+z_1)^2 - x_1x\}}}{2x_1^2 + (y_1+z_1)^2 - 2\sqrt{2}\,x_1\,(y_1+z_1)}.$$

If from any point in any of these curves a straight line be drawn, parallel either to the axis of y, or the axis of z, in direction so as not to meet the right cone, a solution possessing false discontinuity will be obtained. For all such solutions u will be a minimum.

Fourthly, for maximum cases, c_1, c_2, are both negative, and if $c_1 = -\gamma$, $c_2 = -\gamma_2$,

$$\sqrt{2}\,(y+z+A) = 2\sqrt{(\gamma_1+x)(\gamma_2+x)},$$

$$\sqrt{2}\,(y-z+B) = (\gamma_2-\gamma_1)\log\left\{\frac{\gamma_1+\gamma_2+2x}{2} + \sqrt{(\gamma_1+x)\,(\gamma_2+x)}\right\},$$

whence, as before,

$$\sqrt{2}\,(y_1+z_1) = 2\sqrt{(\gamma_1+x_1)(\gamma_2+x_1)} - 2\sqrt{\gamma_1\gamma_2},$$

$$\sqrt{2}\,(y_1-z_1) = (\gamma_2-\gamma_1)\log\frac{\gamma_1+\gamma_2+2x_1+2\sqrt{(\gamma_1+x_1)(\gamma_2+x_1)}}{\gamma_1+\gamma_2+2\sqrt{\gamma_1\gamma_2}}$$

$$= (\gamma_2-\gamma_1)\log\left\{1 + \frac{2x_1+\sqrt{2}\,(y_1+z_1)}{\gamma_1+\gamma_2+2\sqrt{\gamma_1\gamma_2}}\right\}.$$

The same results as before hold when $y_1 = z_1$, pointing to the conjugate character of the then solutions. In general by assuming, in any case, a recognizable relation between γ_1, γ_2, x_1, y_1, z_1, the equation to a surface can be found, and then the equation to the curve from the origin to a point on it, such that u is a maximum or a minimum for that curve, as c_1 is negative or positive, can easily be obtained.

Observe that, always, a line from any point within the cup of the cone drawn parallel to Oy or Oz till it meet the cone, and then drawn straight to the origin, gives a solution as far as the Calculus of Variations is concerned. For such line can, in this calculus, only be compared with other lines passing through the point where it leaves the cone. So such solutions are isolate in character.

There remains for us to investigate the maximorum maxima and minimorum minima. Now if (fig. 67) throughout any finite length of curve, as PQ, p_1, or q_1, is infinite, and if through P_1, Q_1, points on the curve adjacent respectively to P, Q, yet so that p_1 and q_1 are finite throughout PP_1, QQ_1, lines be drawn parallel and equal to PQ, and through their extremities curves be drawn parallel and equal to P_1P, QQ_1, two fresh curves will be obtained throughout which the equation $2p_1q_1 = 1$ holds, and yet for the three curves $\int_{x_0}^{x_1} s\,dx$ is in order of magnitude. So for a maximorum maximum p_1, q_1 cannot be permanently infinite except in the plane $x = x_0$, nor for a minimorum minimum except in the plane $x = x_1$. Also for such portion $\int_{x_0}^{x_0+} s\,dx$ and $\int_{x_1-}^{x_1} s\,dx$ must vanish. So the question is reduced to finding, by the method of Differential Calculus, when, through variation of c_1, c_2,

$$\frac{1}{2}\sqrt{2}\int_0^{x_1}(y+z+A)\,dx = -\int_0^{x_1}\sqrt{(c_1-x)(c_2-x)}\,dx,$$

is a maximum or a minimum. Now, in tracing the surface,

$$\zeta = \int_0^{u_1}\sqrt{(\xi-u)(\eta-u)}\,du,$$

it will be found that there are ridges of true discontinuity. For when ξ is negative, η must be negative, and when ξ and η are both positive, they must be both $> u$, unless $\xi = \eta$. So (fig. 68) the surface has discontinuous ridges indicated by the conditions,

$$\xi = 0,\ \ \eta -;\ \ \eta = 0,\ \ \xi -;$$

$$\xi = u_1,\ \ \eta > u_1;\ \ \eta = u_1,\ \ \xi > u_1;$$

$$\xi = \eta,\ \ \xi > 0 \text{ and } < u_1.$$

Now when c_1, c_2 are both negative,

$$-\int_0^{x_1} \sqrt{(c_1-x)(c_2-x)}\, dx$$

is a minimum when c_1, c_2 both are infinite, and a maximum when both vanish. Likewise, when c_1, c_2 are both positive and $> x_1$, the maximum is when they both $= x_1$, and the minimum when they both are infinite; and, when $c_1 = c_2$, and $< x_1$,

$$-\int_0^{x_1} \sqrt{(c_1-x)^2}\, dx = -\left\{\int_0^{c_1} (c_1-x)\, dx + \int_{c_1}^{x_1} (x-c_1)\, dx\right\}$$

$$= -\left(c_1^2 - \frac{c_1^2}{2} + \frac{x_1^2 - c_1^2}{2} - c_1x_1 + c_1^2\right)$$

$$= -\frac{1}{2}\left(2c_1^2 - 2c_1x_1 + x_1^2\right)$$

$$= -\left(c_1 - \frac{x_1}{2}\right)^2 - \frac{x_1^2}{4},$$

and is a maximum when $x_1 = 2c_1$, and a minimum when $x_1 = 0$, or c_1.

Our conclusion is that the minimorum minimum value of $\int_0^{x_1} s dx$ is given by the curve

$$\sqrt{2}\,\{y - I_1(x)\} = \sqrt{2}\,\{z - I_2(x)\} = x,$$

where $I_1(x)$, $I_2(x)$ vanish when $(x - x_1)$ is not infinitely small, and (being positive),

$$I_1(x_1) = y_1 - \frac{x_1}{\sqrt{2}}, \qquad I_2(x_1) = z_1 - \frac{x_1}{\sqrt{2}};$$

and that the maximorum maximum can be made greater than any assignable quantity, if y_1, z_1 be unrestricted in magnitude; but if y_1, z_1 are given, then the maximorum maximum is given by the curve

$$\sqrt{2}\,\{y - y_1 - I_3(x)\} = \sqrt{2}\,\{z - z_1 - I_4(x)\} = x - x_1,$$

where $I_3(x)$, $I_4(x)$ vanish when x is not infinitely small, and

$$I_3(0) = -y_1, \qquad I_4(0) = -z_1.$$

172. Prob. XIX. To find a curve of given length, such that the area bounded by the curve itself, its two extreme radii of curvature, and the arc of the evolute between them, may be a minimum.

To the enquiry when $\int_0^{s_1} \rho ds$ is a minimum, the answer is clear that it never is. For

$$\int_0^{s_1}\left\{f(s) - a^2 \sin\frac{s\pi}{s_1}\right\} ds \text{ is always } < \int_0^{s_1} f(s)\, ds.$$

But $\int_0^{s_1} \sqrt{\rho^2}\, ds$ is a minimum when $\rho = I(s)$. Among the infinite number of solutions, which satisfy this condition, we may select for a specimen the line whose equation is,

$$\frac{dy}{dx} = \Big|^{a=\infty} \tan^{\frac{1}{3}} ax, \quad y = \Big|^{a=\infty} \frac{1}{a}\int_0^x \tan^{\frac{1}{3}} ax dx;$$

observe that

$$\int_{\frac{\pi}{4a}}^{\frac{\pi}{2a}} \tan^{\frac{1}{3}} ax dx < \frac{1}{a \sin\frac{\pi}{4}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin\phi d\phi}{\cos^{\frac{1}{3}}\phi},$$

and is therefore finite. The value of ρ in this curve is,

$$\Big|^{a=\infty} \frac{3}{a}\left(\cos^{\frac{2}{3}} ax + \sin^{\frac{2}{3}} ax\right)^{\frac{3}{2}} \cos^{\frac{1}{3}} ax \sin^{\frac{2}{3}} ax,$$

which vanishes when $\frac{dy}{dx}$ is infinite.

But if we introduce the restriction that the curve joins two given points, at a given distance, not greater than the given length, from each other, we might proceed by discussing the integral

$$\int_0^{s_1}\left\{\rho + \lambda\frac{d(r^2)}{ds}\right\} ds,$$

where $$\rho\left\{2 - \frac{d^2(r^2)}{ds^2}\right\} + 2\left\{r^2 - \frac{d(r^2)^2}{ds}\right\}^{\frac{1}{2}} = 0;$$

or, if we restrict by supposing the extreme points of the curve given in position, we may investigate when

$$u \equiv \int_{x_0}^{x_1} \left\{ \frac{(1+p_1^2)^2}{p_2} + \lambda p_1 + \mu \sqrt{1+p_1^2} \right\} dx$$

is a minimum. If in our solution of this, p_2 throughout is positive, we shall therein satisfy the problem, as appears from the coefficient of δp^2 in the argument of the integral $\delta^2 u$. We have

$$\delta u \equiv \int_{x_0}^{x_1} \left[\left\{ \frac{4p_1(1+p_1^2)}{p_2} + \lambda + \frac{\mu p_1}{\sqrt{1+p_1^2}} \right\} \delta p_1 - \frac{(1+p_1^2)^2}{p_2^2} \delta p_2 \right] dx,$$

$$\delta^2 u \equiv \int_{x_0}^{x_1} \left[\left\{ \frac{4+12p_1^2}{p_2} + \frac{\mu}{(1+p_1^2)^{\frac{3}{2}}} \right\} \delta p_1^2 \right.$$

$$\left. - \frac{8p_1(1+p_1^2)}{p_2^2} \delta p_1 \delta p_2 + \frac{2(1+p_1^2)^2}{p_2^3} \delta p_2^2 \right] dx.$$

Also

$$-\frac{(1+p_1^2)^2}{p_2^2} - \frac{d^{-1}}{dx^{-1}} \left\{ \frac{4p_1(1+p_1^2)}{p_2} + \lambda + \frac{\mu p_1}{\sqrt{1+p_1^2}} \right\} = I(x).$$

Whence

$$\frac{2(1+p_1^2)^2 p_3}{p_2^3} - \frac{8(1+p_1^2)p_1}{p_2} + \lambda + \frac{\mu p_1}{\sqrt{1+p_1^2}} = \frac{dI}{dx},$$

$$-\frac{2(1+p_1^2)^2}{p_2} + \lambda p_1 + \mu \sqrt{1+p_1^2} = \int p_2 \frac{dI}{dx} dx.$$

And, during an interval throughout which $p_2 \dfrac{dI}{dx}$ vanishes,

$$(\lambda p_1 + \mu \sqrt{1+p_1^2} - b)\, p_2 = 2(1+p_1^2)^2,$$

or
$$\frac{dx}{dp_1} = \frac{\lambda p_1 - b + \mu \sqrt{1+p_1^2}}{2(1+p_1^2)^2},$$

as p_1 is given if x is given, but x may have many values for the same value of p_1, in integrating this the constant introduced must be capable of being truly discontinuous. (Just as if the relation, $y = \tan(x - x_0)$, is the one to be obtained from $\frac{dx}{dy} = \frac{1}{1+y^2}$, the constant introduced must be truly discontinuous, if the range for $(x - x_0)$ exceeds an integral multiple of π.) We have also

$$\frac{dy}{dp_1} = \frac{\lambda p_1^2 - bp_1 + \mu p_1 \sqrt{1+p_1^2}}{2(1+p_1^2)^2};$$

so that

$$x - h = -\frac{b}{4}\tan^{-1} p_1 - \frac{bp_1 + \lambda - 2\mu p_1\sqrt{1+p_1^2}}{4(1+p_1^2)},$$

$$y - k = \frac{\lambda}{4}\tan^{-1} p_1 + \frac{-\lambda p_1 + b - 2\mu\sqrt{1+p_1^2}}{4(1+p_1^2)}.$$

Again $$s = \int_0^x \frac{ds}{dx}\,dx = \int_0^{p_1} \frac{\lambda p_1 - b + \mu\sqrt{1+p_1^2}}{2(1+p_1^2)^{\frac{3}{2}}}\,dp_1$$

$$= \Big|_0^{p_1} \left\{ -\frac{\lambda + bp_1}{2(1+p_1^2)^{\frac{1}{2}}} + \frac{\mu}{2}\tan^{-1} p_1 \right\}.$$

We obtain in these six equations connecting x_0, y_0, s_0, x_1, y_1, s_1, h, k, b, λ, μ, $\Big|^{x=x_0} p_1$, $\Big|^{x=x_1} p_1$, whereof the six first are given. As there results only two equations connecting b, λ, μ, we have an infinite number of solutions, all of which render u a minimum, (except when, μ being negative, $\delta^2 u$ can be made negative, or to vanish,) and among these there will be minima minimorum. We proceed to investigate them.

$$u \equiv \int_{x_0}^{x_1} \left\{ \frac{(1+p_1^2)^2}{p_2} + \lambda p_1 + \mu\sqrt{1+p_1^2} \right\} dx$$

$$= \Big|_{x=x_0}^{x=x_1} \int \left(\frac{3\lambda p_1 + 3\mu\sqrt{1+p_1^2} - b}{2} \right) \frac{\lambda p_1 - b + \mu\sqrt{1+p_1^2}}{2(1+p_1^2)^2}\,dp_1$$

$$= \Big|_{x=x_0}^{x=x_1} \int \frac{3(\lambda^2+\mu^2)p_1^2 - 4\lambda b p_1 + 3\mu^2 + b^2 + (6\lambda\mu p_1 - 4\mu b)\sqrt{1+p_1^2}}{4(1+p_1^2)^2}\,dp_1$$

$$= \Bigg|_{x=x_0}^{x=x_1} \int \left\{ \frac{-b\,(\lambda p_1 - b + \mu\sqrt{1+p_1^2})}{4\,(1+p_1^2)^2} + \frac{3\lambda\,(\lambda p_1^2 - bp_1 + \mu p_1\sqrt{1+p_1^2})}{4\,(1+p_1^2)^2} \right.$$

$$\left. + \frac{3\mu^2 p_1^2 + 3\mu^2 + 3\mu\,(\lambda p_1 - b)\sqrt{1+p_1^2}}{4\,(1+p_1^2)^2} \right\} dp_1$$

$$= \frac{-b\,(x_1 - x_0) + 3\lambda\,(y_1 - y_0) + 3\mu\,(s_1 - s_0)}{2}.$$

This has to be made a minimum, b, λ, μ, varying. Now, wherever h, k, b, λ, μ, occur they are linear, and so two equations can be obtained connecting u, $\Big|^{x=x_0} p_1$, $\Big|^{x=x_1} p_1$. The determination of the minima values of u from these equations will essentially be difficult. For it is clear that an infinite number of functions can be assigned, which will make $\{u - \int_{x_0}^{x_1} (\lambda p_1 + \mu\sqrt{1+p_1^2})\,dx\}$ vanish, even when p_2 is restricted to being positive, a condition requisite if there is to be a minimum minimorum at all.

If $\lambda = \mu = 0$, the restriction, that $\int_{x_0}^{x_1} p_1 dx$, $\int_{x_0}^{x_1} \sqrt{1+p_1^2}\,dx$, be constant, is removed. We then have

$$x - h = -\frac{b}{4}\tan^{-1} p_1 - \frac{bp_1}{4\,(1+p_1^2)},$$

$$y - k = \frac{b}{4\,(1+p_1^2)},$$

the equations to a cycloid. The possibility of $\frac{dI}{dx}$ being instantaneously infinite, shows that b may be replaced by a function $W(x)$. The geometrical interpretation of this is, that any succession of cycloids, arranged so as to form a continuous curve, the axes of the cycloids being parallel, and the radius of the generating circle for each perfectly arbitrary, solves the problem. It is easily seen that the minimum minimorum in this case requires each cycloid to be indefinitely small. It will have for equations,

$$y = k, \quad x - h = \Bigg|^{b=0} \left\{ \tan^{-1} p_1 + \frac{p_1}{1+p_1^2} \right\} \frac{b}{4}.$$

173. Prob. XX. Having given the conditions,

$$\frac{dF}{dy}p_1 + \frac{dF}{dz}q_1 + \frac{dF}{dx} = 0,$$

$$\int_{x_0}^{x_1} p_1 dx = y_1 - y_0, \quad \int_{x_0}^{x_1} q_1 dx = z_1 - z_0,$$

connecting x, y, z, p_1, q_1, to find when $\int_{x_0}^{x_1}\sqrt{1+p_1^2+q_1^2}\,dx$ is a minimum.

This is the analytical problem to which the geometrical problem of finding the shortest line on a surface leads. The solution will indicate the character which all geodesic lines possess. But no curve will be embraced in the solution of the analytical question for which s has more than one value when x is given.

We have to make a minimum,

$$u \equiv \int_{x_0}^{x_1}\left\{\sqrt{1+p_1^2+q_1^2} + X\left(\frac{dF}{dy}p_1 + \frac{dF}{dz}q_1 + \frac{dF}{dx}\right) + \lambda p_1 + \mu q_1\right\} dx,$$

where X may involve x, but λ, μ are constants. We have

$$\begin{aligned}\delta u \equiv \int_{x_0}^{x_1}\Bigg\{&\left(\frac{p_1}{\sqrt{1+p_1^2+q_1^2}} + X\frac{dF}{dy} + \lambda\right)\delta p_1\\ &+\left(\frac{q_1}{\sqrt{1+p_1^2+q_1^2}} + X\frac{dF}{dz} + \mu\right)\delta q_1\\ &+X\left(\frac{d^2F}{dy^2}p_1 + \frac{d^2F}{dydz}q_1 + \frac{d^2F}{dxdy}\right)\delta y\\ &+X\left(\frac{d^2F}{dydz}p_1 + \frac{d^2F}{dz^2}q_1 + \frac{d^2F}{dxdz}\right)\delta z\Bigg\},\end{aligned}$$

$$\delta^2 u \equiv \int_{x_0}^{x_1}\left\{\frac{(1+q_1^2)\,\delta p_1^2 - 2p_1q_1\delta p_1\delta q_1 + (1+p_1^2)\,\delta q_1^2}{(1+p_1^2+q_1^2)^{\frac{3}{2}}} + \dots\right\} dx,$$

the other terms in $\delta^2 u$ we need not concern ourselves with. $\delta^2 u$ may vanish for the particular variation which passes from one geodesic to another; otherwise it must be positive, for

$$\delta p_1^2 + \delta q_1^2 + (q_1 \delta p_1 - p_1 \delta q_1)^2 \text{ is } +.$$

So
$$\frac{p_1}{\sqrt{1+p_1^2+q_1^2}} + X\frac{dF}{dy} + \lambda$$
$$- \frac{d^{-1}}{dx^{-1}}\left\{X\left(\frac{d^2F}{dy^2}p_1 + \frac{d^2F}{dydz}q_1 + \frac{d^2F}{dxdy}\right)\right\} = I_y(x),$$

whence
$$\frac{d}{dx}\frac{p_1}{\sqrt{1+p_1^2+q_1^2}} + \frac{dX}{dx}\frac{dF}{dy} = \frac{dI_y}{dx}.$$

Similarly,
$$\frac{d}{dx}\frac{q_1}{\sqrt{1+p_1^2+q_1^2}} + \frac{dX}{dx}\frac{dF}{dz} = \frac{dI_z}{dx};$$

$$\therefore \frac{dF}{dz}\frac{d}{dx}\frac{p_1}{\sqrt{1+p_1^2+q_1^2}} - \frac{dF}{dy}\frac{d}{dx}\frac{q_1}{\sqrt{1+p_1^2+q_1^2}} = \frac{dF}{dz}\frac{dI_y}{dx} - \frac{dF}{dy}\frac{dI_z}{dx}.$$

Through an interval, throughout which this vanishes, this will be satisfied by any curve which consists of portions of geodesic curves, finite in number. The variation which will pass from one such curve to another indefinitely near to it will make $\delta^2 u$ vanish. The sharp angles may be rounded off by using Fourierian definite integrals to express the equation to such curve. When $\frac{dF}{dz}\frac{dI_y}{dx} - \frac{dF}{dy}\frac{dI_z}{dx}$ is finite, a finite portion of the curve lies in a plane parallel to plane of yz. All curves having any such finite portions so lying can only, in Variations Calculus, be compared with curves coinciding with them throughout such finite portions.

NOTE on 149. Prob. II. If the biquadratic in λ has no real root, u cannot be made a maximum or a minimum. This is evident from our solution, or may be made evident as follows.

If possible, let $y =$ some not truly discontinuous function of x, render u a maximum or a minimum: then, (except through intervals whose sum altogether is infinitely small,) either $(ax - 3y^2)$ must be constant, from $x = x_0$ till $x = x_1$, or not. If $(ax - 3y^2)$ is so constant, and y does not change sign,

$$\sqrt{c^3} = \int_{x_0}^{x_1} \sqrt{\frac{ax + \lambda'}{3}}\, dx, \text{ cannot be less than } 2\left\{\frac{a^{\frac{1}{3}}(x_1 - x_0)}{3}\right\}^{\frac{3}{2}},$$

(a is supposed +,) but may have any greater value: in this case the biquadratic will always have a real root. If y does change sign, as it will if $3y^2 = ax + I(x)$, and y change sign sometimes when y^2 vanishes, variations applied only when y is + diminish u, and only when y is negative increase u. If $(ax - 3y^2)$ is not so constant, it must have, owing to continuity, its mean value, λ_0, for some value of x, and there will be two finite intervals such that $(ax - 3y^2 - \lambda_0)$ is + throughout one, and − throughout the other. Take $\delta\tau$ always + throughout these two intervals and zero elsewhere, and when $(ax - 3y^2 - \lambda_0)$ vanishes. The value of $\delta\tau$ in each interval can be varied without changing its value in the other, and so,

$$\int_{x_0}^{x_1} \delta y dx = \int_{x_0}^{x_1} \frac{\delta\tau dx}{ax - 3y^2 - \lambda_0},$$

can be made to vanish without $\int_{x_0}^{x_1} \delta\tau dx$ vanishing. Now

$$\delta u = \int_{x_0}^{x_1} \{(ax - 3y^2 - \lambda_0)\,\delta y + 3y\delta y^2 - \delta y^3\}\, dx$$

$$= \int_{x_0}^{x_1} \delta\tau dx + \int_{x_0}^{x_1} (3y\delta y^2 - \delta y^3)\, dx,$$

and is initially +, since y is always finite. In like manner δu can be made negative.

Thus it can be shown, in all cases where Taylor's Theo does not fail, that there can be no maximum or minimum un δu vanish. When however, as in Prob. VI., $I(x)$ can be chosen that y pass from one form to another of those wh render u a minimum, of the infinite number of solutions the mi mum minimorum will be the sky-line solution there indicat Now by a proper choice of $I(x)$, a falsely discontinuous functi can be assigned for y so that u have the same value as when has the truly discontinuous form which it has in the sky-line.

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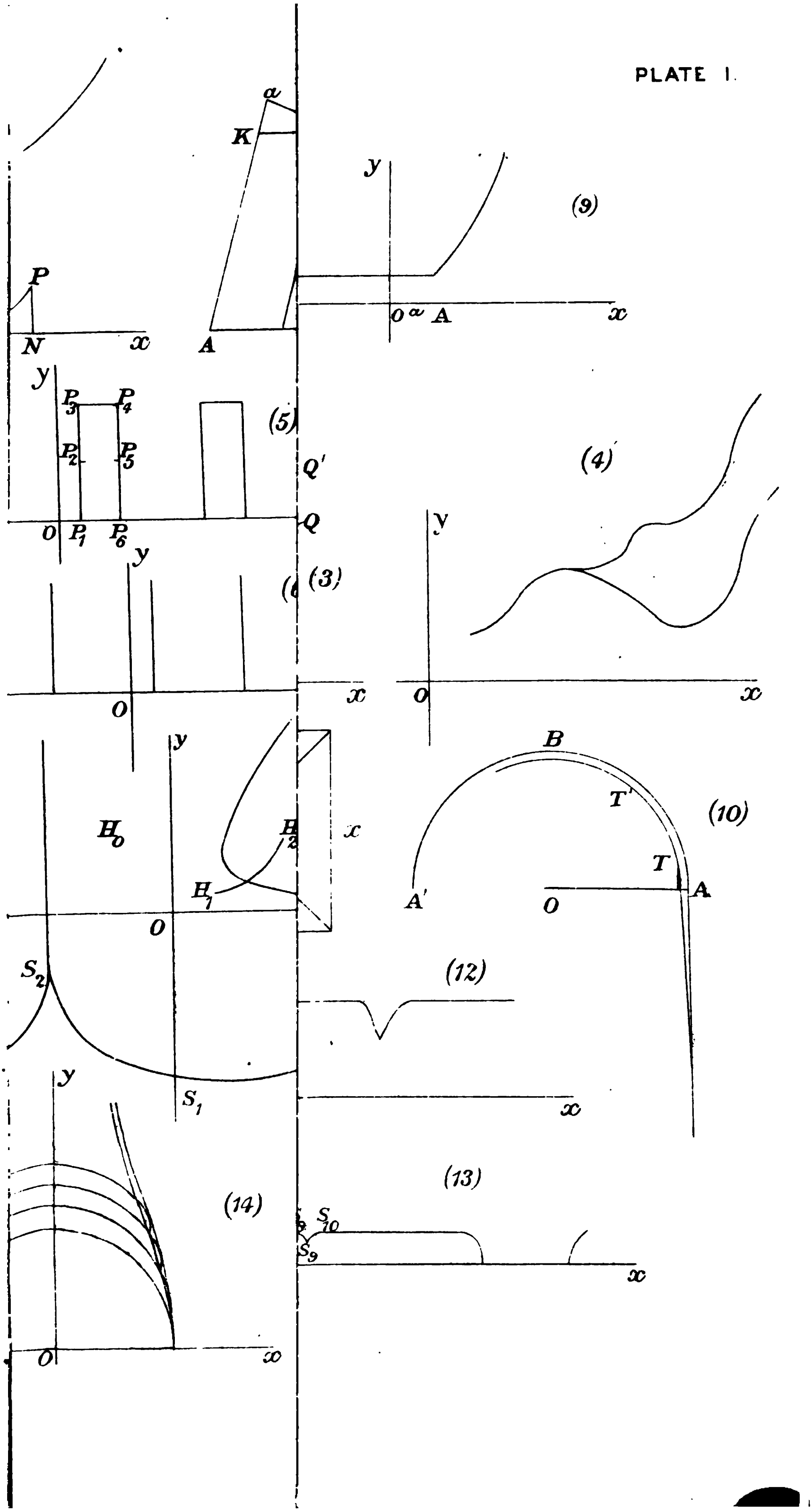

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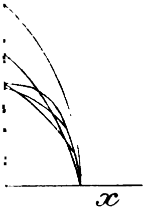
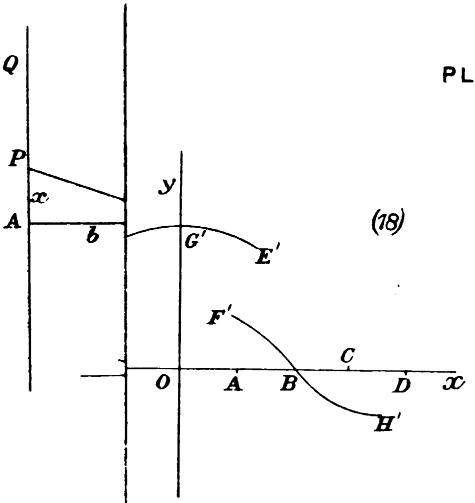
PLATE 2.

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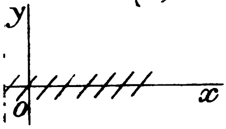
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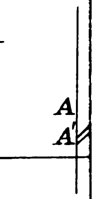

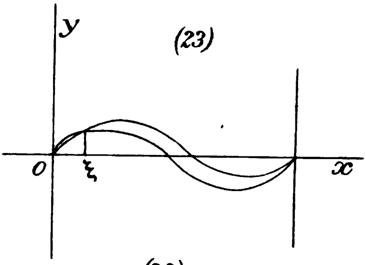
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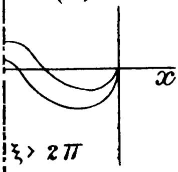
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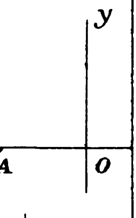

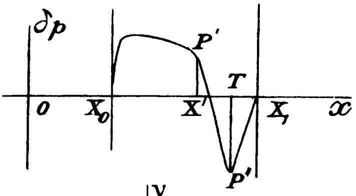
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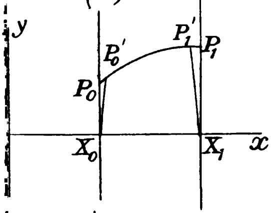
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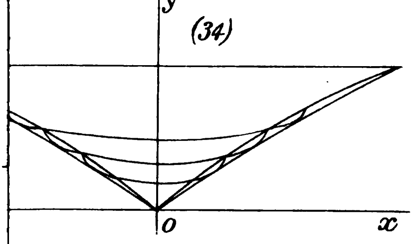
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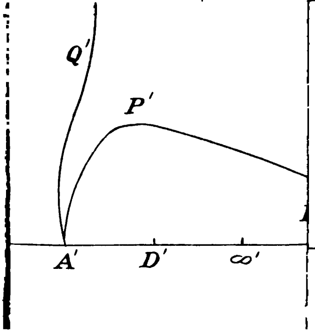

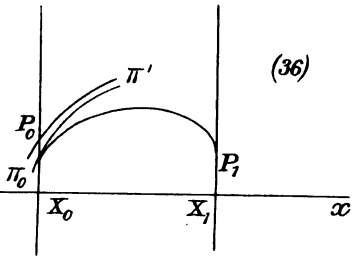
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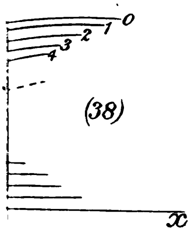

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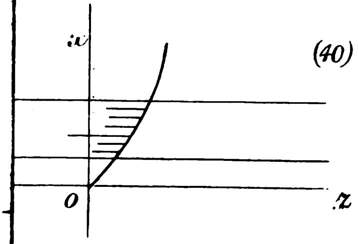

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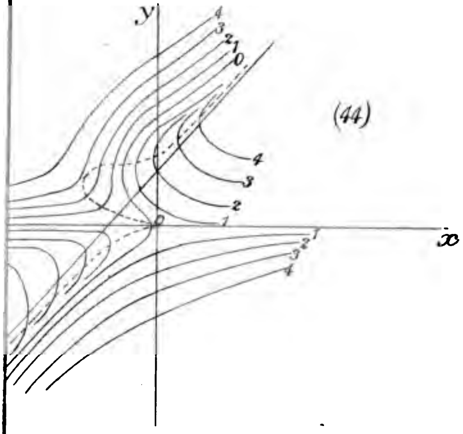

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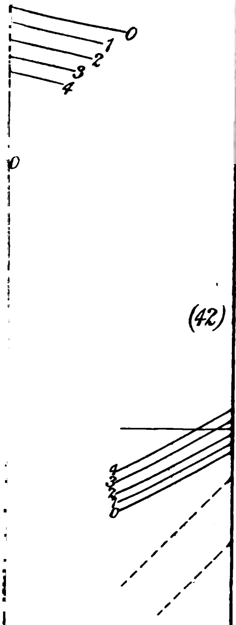

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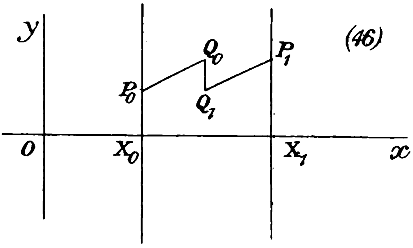

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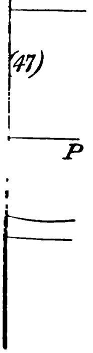

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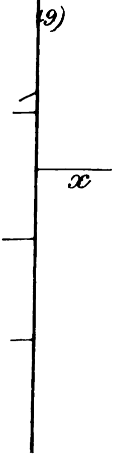

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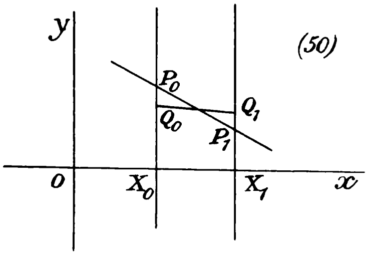

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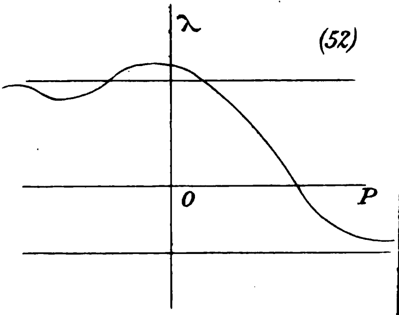
PLATE 4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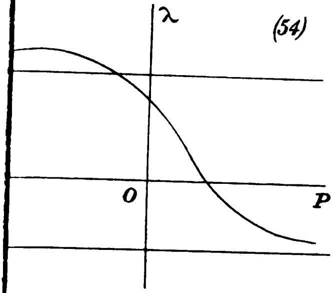
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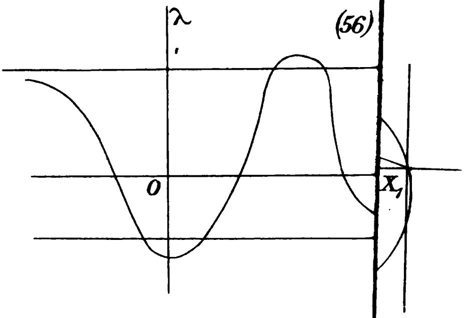
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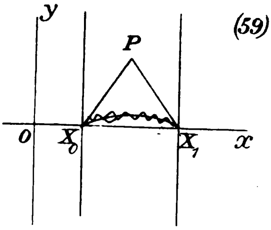
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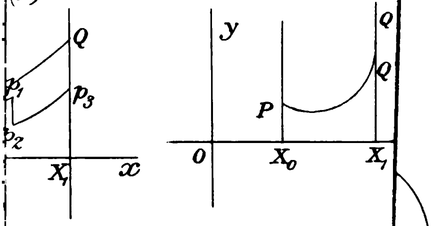

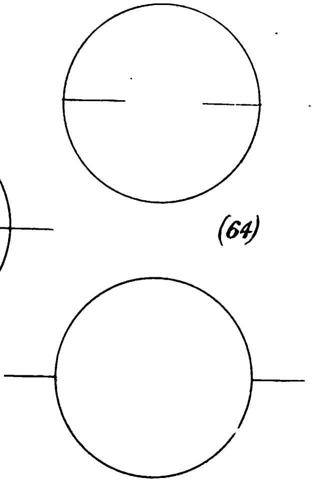
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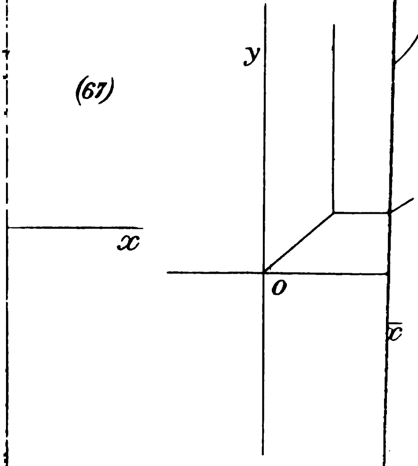

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